Matrices

What is a Matrix?

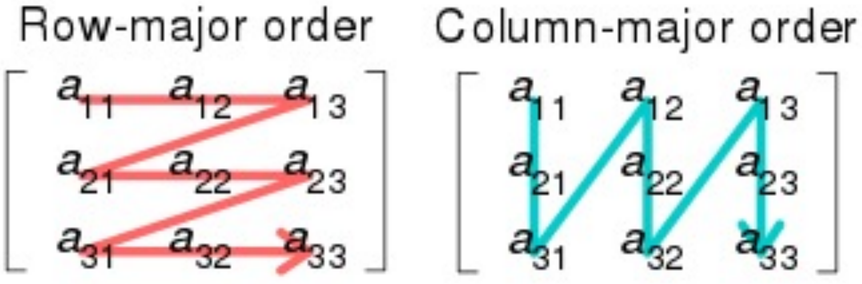
- For us, a collection of scalar values arranged in a multi-dimensional array (grid)
- The dimension of a matrix is usually represented as m x n (m-by-n) where m is the number or rows, and n is the number of columns
- In computer graphics we deal mostly with square matrices of dimensions:
  - 2x2, 3x3, and 4x4

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Matrix Operations - Transpose

- Sometimes we need to change the order of the values in the matrix
- We may do this as it is part of an operational formulation
- It may also be necessary to provide the correct representation of the data to WebGL which take column major order data

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$



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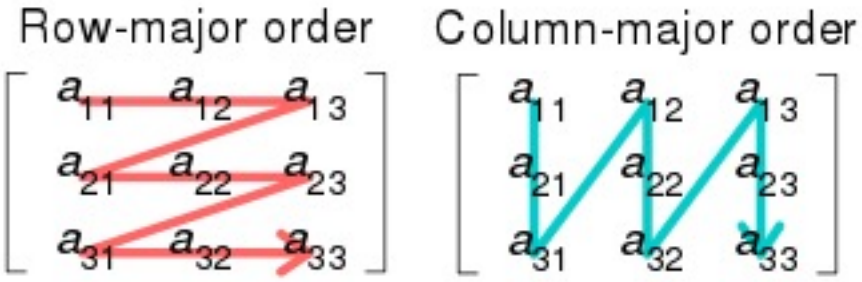

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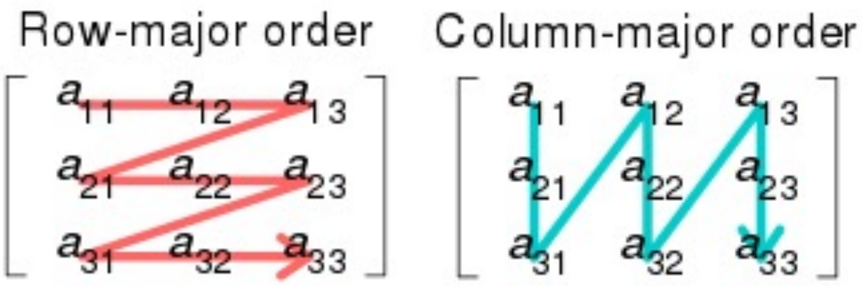

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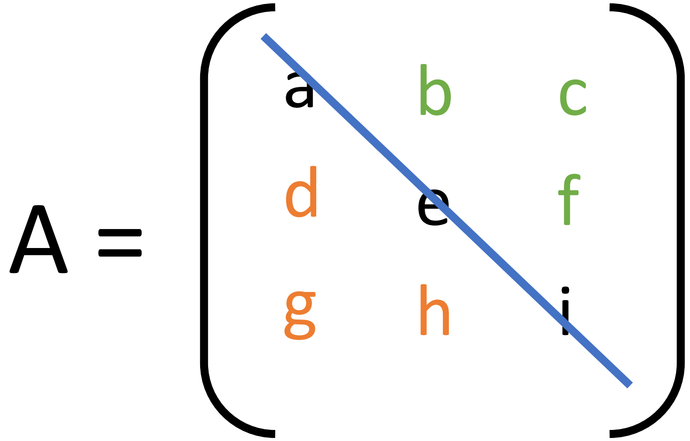

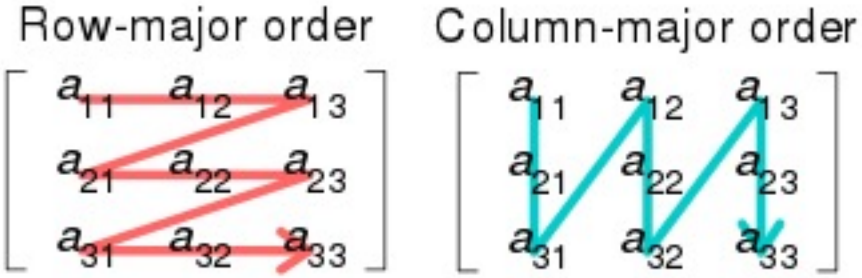

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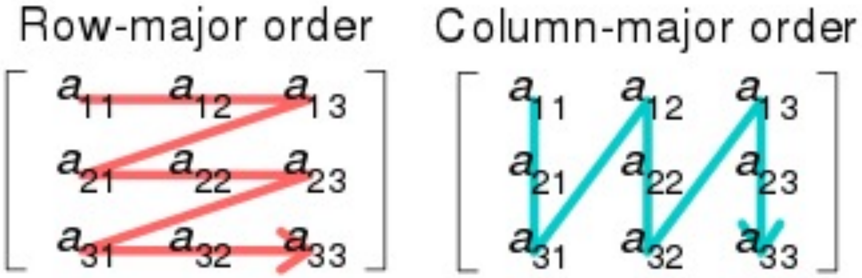

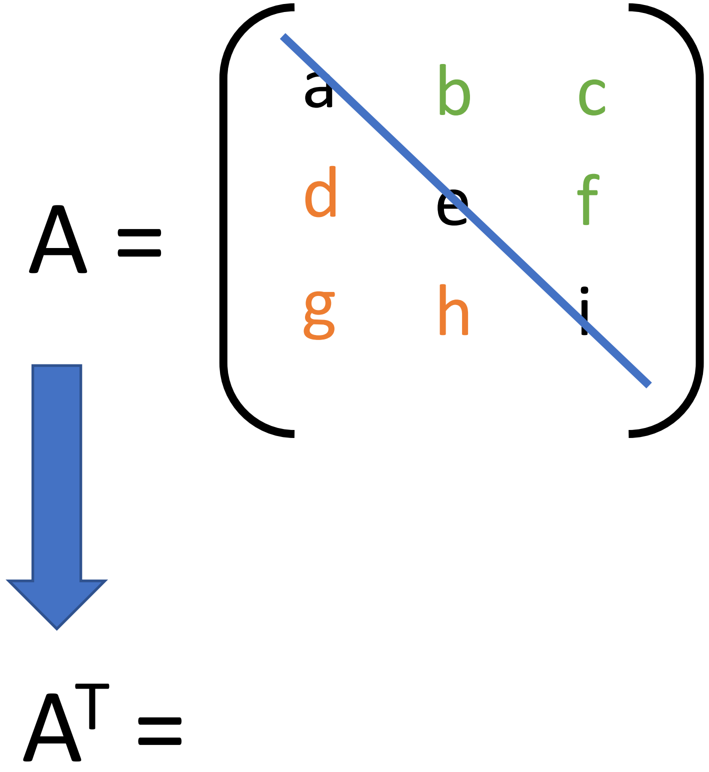

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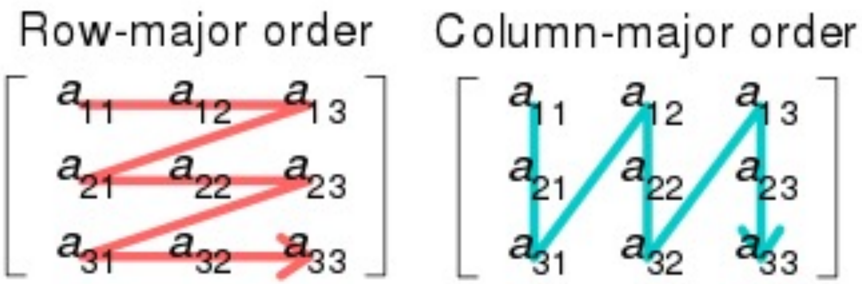

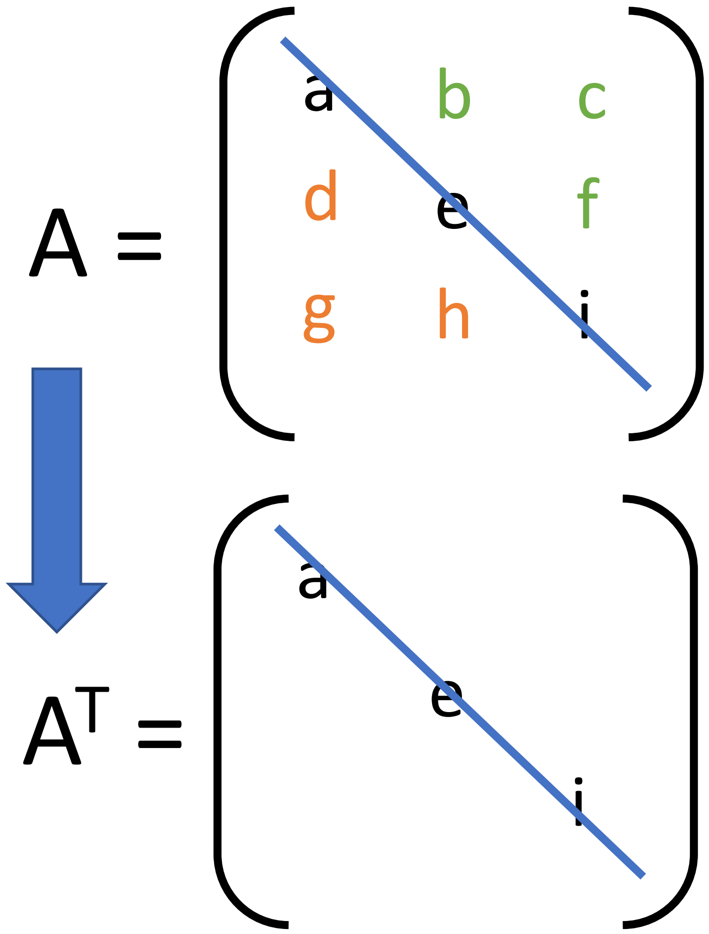

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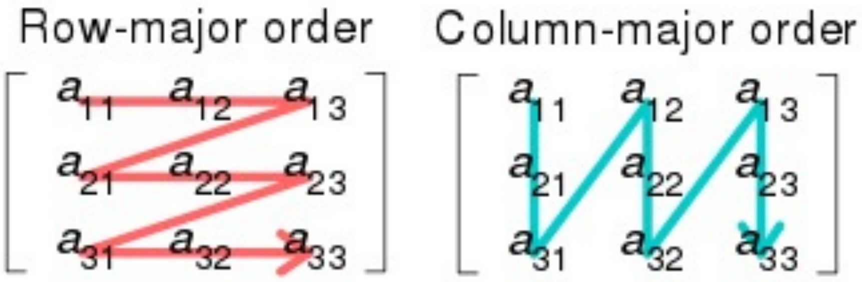

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

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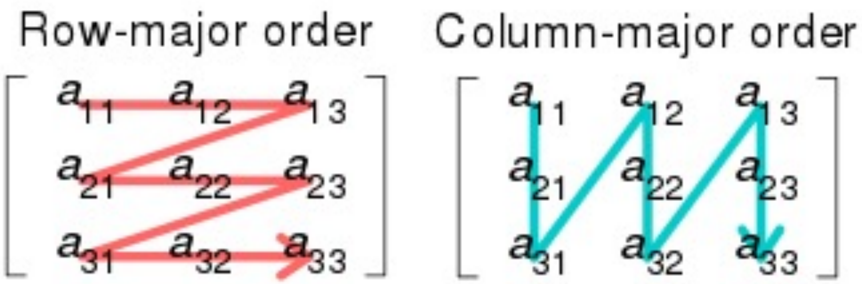

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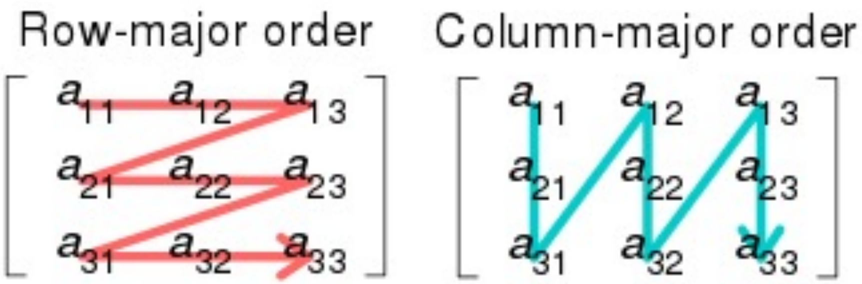

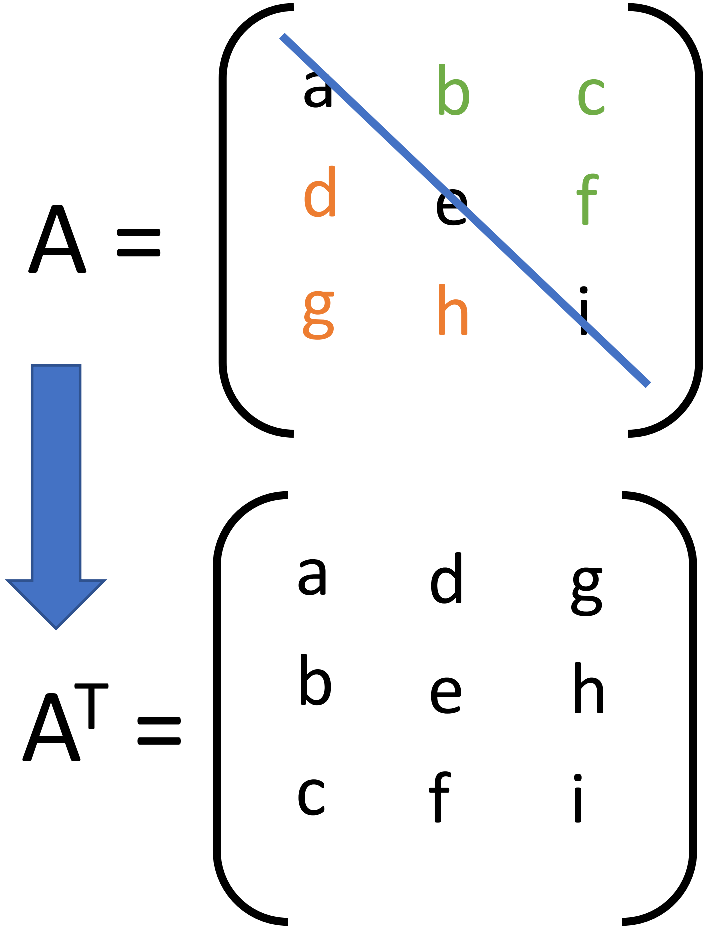

Matrix Operations - Inverse

- How to calculate the inverse of a matrix by hand is out of the scope of this class
  - If you are curious (Gauss-Jordan Method)
  - Online Calculator

- MV.js provides a function called ***inverse***

- **Be careful!** If the determinant of the matrix is zero, there is **NO** inverse
  - MV.js provides a determinant function called ***det***

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

$$A' = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

Matrix Operations – Matrix Addition

- A matrix can be added to another matrix if the number of rows and columns are the same for both matrices

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \quad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

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2x2 2x2

Matrix Operations – Matrix Addition

- A matrix can be added to another matrix if the number of rows and columns are the same for both matrices
- Values at matching element positions are added with each other

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \quad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \quad A + B = \begin{pmatrix} a_{00}+b_{00} & a_{01}+b_{01} \\ a_{10}+b_{10} & a_{11}+b_{11} \end{pmatrix}$$

2x2 2x2

Matrix Operations – Scalar Multiplication

- A matrix can be multiplied by a single scalar value

$\alpha = 5$

$$B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

Matrix Operations – Scalar Multiplication

- A matrix can be multiplied by a single scalar value
- For this we simply multiply each element in the matrix by the scalar

$$\alpha = 5 \qquad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

$$\alpha B = \begin{pmatrix} 5b_{00} & 5b_{01} \\ 5b_{10} & 5b_{11} \end{pmatrix}$$

Matrix Operations – Matrix Multiplication

- A matrix can be multiplied by another matrix or a vector if the number of rows in the second vector or matrix are equal to the number of columns in the first matrix

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \qquad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

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Multiply?

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2x2 YES! 2x2

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Multiply?

2x2 NO! 1x2

Matrix Operations - Multiplication

- A matrix can be multiplied by another matrix or a vector if the number of rows in the second vector or matrix are equal to the number of columns in the first matrix
- When you multiply two matrices or a matrix and a vector, you can determine the output dimensions of the product

A = 2x2   B = 2x2

A B = ??

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A = 3x3   B = 3x1

A B = ??

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Matrix Operations - Multiplication

- A matrix can be multiplied by another matrix or a vector if the number of rows in the second vector or matrix are equal to the number of columns in the first matrix
- When you multiply two matrices or a matrix and a vector, you can determine the output dimensions of the product
- **MATRIX MULTIPLICATION IS NOT COMMUTATIVE**

A = 3x3   B = 3x1

A B != B A

In this case not possible

A = 3x3   B = 3x3

A B != B A

In this case different answer

Matrix Multiplication Example

- Multiplication is done using the rows of the multiplicand (left hand) by the columns of the multiplier (right hand)

$$A\,B = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

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Matrix Multiplication Example

- Multiplication is done using the rows of the multiplicand (left hand) by the columns of the multiplier (right hand)

$$AB = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} = \begin{pmatrix} a_{00}*b_{00} + a_{01}*b_{10} & a_{00}*b_{01} + a_{01}*b_{11} \\ a_{10}*b_{00} + a_{11}*b_{10} & \end{pmatrix}$$

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- Multiplication is done using the rows of the multiplicand (left hand) by the columns of the multiplier (right hand)

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The Identity Matrix

- A square mxn (m-by-n) matrix where all the values along the diagonal are 1's and the rest are 0's.
- Usually shown as a capital **I**
- When used in multiplication it is like multiplying a scalar by 1
- If A is an mxn matrix:
  - $\mathbf{I}_m A = A\mathbf{I}_n = A$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Why Matrices in CG?

- Matrices allow for transformations of the vectors
- This includes the ability to change coordinate systems

Changing Coordinate Systems

- Suppose we have a vector *w* with the following column matrix representation in our basis

$$w = x + 2y + 3z$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{matrix} \text{x component} \\ \text{y component} \\ \text{z component} \end{matrix}$$

Changing Coordinate Systems

- Suppose we have a vector *w* with the following column matrix representation in our basis
- Suppose we need to convert *w* to a different system x', y', z' defined by:

$x' = x$

$y' = x + y$

$z' = x + y + z$

$w = x + 2y + 3z$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

x component

y component

z component

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$$\begin{aligned} x' &= x \\ y' &= x + y \\ z' &= x + y + z \end{aligned} \quad \Rightarrow \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

- To create the transformation matrix (T), we take the transpose of the matrix and then the inverse of the transpose $(M^T)^{-1}$

Changing Coordinate Systems

- The resulting matrix is now:

$$w = x + 2y + 3z$$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = T$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{matrix} \text{x component} \\ \text{y component} \\ \text{z component} \end{matrix}$$

- The new representation of *w* is expressed by Ta:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \qquad w = -x - y + 3z$$

Changing Coordinate Systems

- What is interesting to note about T is that the columns represent the original x, y, and z in the new coordinate system x', y', z'.

$$T = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

What about Homogeneous Coordinates?

- Like before, we have a series of basis vectors but also a point.
- Assume new basis vectors and the reference point will also change:

$$\left.\begin{aligned} x' &= x \\ y' &= x + y \\ z' &= x + y + z \end{aligned}\right\} \Rightarrow M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

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- Like before, we have a series of basis vectors but also a point.
- Assume new basis vectors and the reference point will also change:

$$x' = x$$
$$y' = x + y$$
$$z' = x + y + z$$
$$Q_0 = x + 2y + 3z + P_0$$

$$\Rightarrow M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

What about Homogeneous Coordinates?

- Like before, we have a series of basis vectors but also a point.
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- Remember that we are interested in the Transpose of the matrix

$x' = x$

$y' = x + y$

$z' = x + y + z$

$Q_0 = x + 2y + 3z + P_0$

$$M^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What about Homogeneous Coordinates?

- Like before, we have a series of basis vectors but also a point.
- Assume new basis vectors and the reference point will also change
- Remember that we are interested in the Transpose of the matrix
- Last, we need the inverse of M^T

$$\left.\begin{aligned} x' &= x \\ y' &= x + y \\ z' &= x + y + z \\ Q_0 &= x + 2y + 3z + P_0 \end{aligned}\right\} \Rightarrow (M^T)^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What about Homogeneous Coordinates?

- Like before, we have a series of basis vectors but also a point.
- Assume new basis vectors and the reference point will also change
- Remember that we are interested in the Transpose of the matrix
- Last, we need the inverse of M^T
- Now we can convert P_0 to Q_0 ($Q_0 = x + 2y + 3z + P_0$)

$$(M^T)^{-1}P_0 = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = Q_0$$

Switching Coordinate Frames Easily

$$(M^T)^{-1}P_0 = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = Q_0$$

$$((M^T)^{-1})Q_0 = M^T Q_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = P_0$$

Switching Coordinate Frames Easily

Points

$$(M^T)^{-1}P_0 = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = Q_0$$

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Switching Coordinate Frames Easily

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Points

Vector component of points