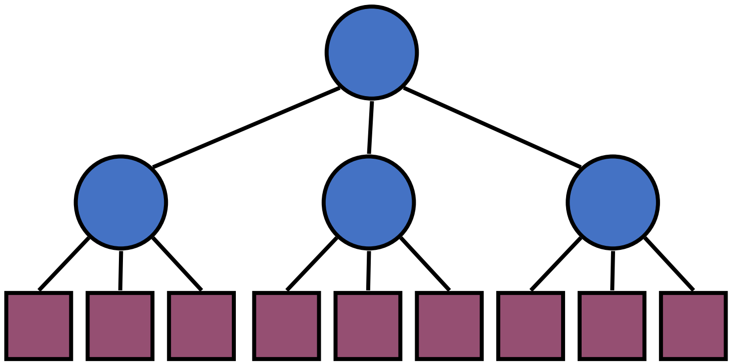
Divide and Conquer

CLRS 4.1 – 4.5

(+ some supplemental material)

Recap

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data in two or more disjoint subsets S1, S2, …
  - Conquer: solve the subproblems recursively
  - Combine: combine the solutions for S1, S2, …, into a solution for S
- The base case for the recursion are subproblems of constant size
- Analysis can be done using recurrence equations



Recall: Merge Sort

Merge sort works on an input sequence with *n* elements and consists of 3 steps:

- Divide: partition the *n*-element sequence to be sorted into two subsequences of *n/2* elements each
- Conquer: sort the two subsequences recursively using merge sort
- Combine: merge the two sorted subsequences to produce the sorted answer

```
MERGE-SORT(A, p, r)
if p < r
    q = ⌊(p + r)/2⌋
    MERGE-SORT(A, p, q)
    MERGE-SORT(A, q + 1, r)
    MERGE(A, p, q, r)
```

Recall: Merge Sort – Recurrence Equation Analysis

- The conquer step of merge-sort consists of merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes at most bn steps, for some constant b.
- Likewise, the basis case ($n < 2$) will take at most b steps.
- Therefore, if we let $T(n)$ denote the running time of merge-sort:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

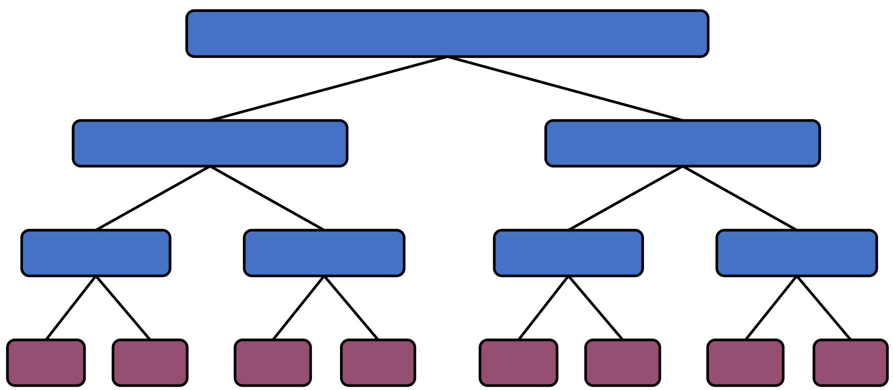
- We can analyze the running time of merge-sort by finding a closed form solution to the above equation.
  - That is, a solution that has $T(n)$ only on the left-hand side.

Recurrence Equation: Recursion Tree

Draw the recursion tree for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

depth	T's	size	time
0	1	$\boldsymbol{n}$	$\boldsymbol{bn}$
1	2	$\boldsymbol{n}/2$	$\boldsymbol{bn}$
$\boldsymbol{i}$	2^i	$\boldsymbol{n}/2^i$	$\boldsymbol{bn}$
…	…	…	…



Total time = $\boldsymbol{bn} + \boldsymbol{bn} \log \boldsymbol{n}$

(last level plus all previous levels)

Recurrence Equation: Iterative Substitution

In the iterative substitution, or "plug-and-chug," technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern, then prove it is true by induction:

$$
\begin{aligned}
T(n) &= 2T(n/2) + bn \\
&= 2(2T(n/2^2)) + b(n/2)) + bn \\
&= 2^2 T(n/2^2) + 2bn \\
&= 2^3 T(n/2^3) + 3bn \\
&= 2^4 T(n/2^4) + 4bn \\
&= \ldots \\
&= 2^i T(n/2^i) + ibn
\end{aligned}
$$

- Note that the base case, $T(n) = b$, case occurs when $2^i = n$. That is, $i = \log n$. So we have:
- **Once we prove this by induction**, then $T(n)$ is $O(n\log n)$.

Recurrence Equation: Guess-and-Test Method (1)

In the guess-and-test method, we guess a closed form solution and then **try to prove it is true by induction:**

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

- Guess #1: $T(n) \leq cn\log n$.

$$\begin{aligned} T(n) &= 2T(n/2) + bn\log n \\ &\leq 2(c(n/2)\log(n/2)) + bn\log n \\ &= cn(\log n - \log 2) + bn\log n \\ &= cn\log n - cn + bn\log n \end{aligned}$$

- Wrong: we cannot make this last line be less than $cn\log n$

Recurrence Equation: Guess-and-Test Method (2)

Recall the recurrence equation:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

- <u>Guess #2</u>: T(n) ≤ $cn\log^2 n$.

$$\begin{aligned} T(n) &= 2T(n/2) + bn\log n \\ &\leq 2(c(n/2)\log^2(n/2)) + bn\log n \\ &= cn(\log n - \log 2)^2 + bn\log n \\ &= cn\log^2 n - 2cn\log n + cn + bn\log n \\ &\leq cn\log^2 n \end{aligned}$$

- So, T(n) is $O(n\log^2 n)$.

In general, to use this method, you need to have a good guess.

Recurrence Equation: Master Method

Many divide-and-conquer recurrence equations have the form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

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Master method: Ex 1

Form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$

Master Theorem:

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3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.

Example: $T(n) = 4T(n/2) + n$

Solution: $\log_b a = 2$, so case 1 says T(n) is $\Theta(n^2)$.

Master method: Ex 2

Form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$

Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.

Example: $T(n) = 2T(n/2) + n\log n$

Solution: $\log_b a=1$, so case 2 says T(n) is $\Theta(n \log^2 n)$.

Master method: Ex 3

Form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$

Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.

Example: $T(n) = T(n/3) + n \log n$

Solution: $\log_b a=0$, so case 3 says T(n) is Θ(n log n).

Master method: Ex 4

Form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$

Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.

Example: $T(n) = 8T(n/2) + n^2$

Solution: $\log_b a=3$, so case 1 says T(n) is $\Theta(n^3)$.

Master method: Ex 5

Form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example: $T(n) = 9T(n/3) + n^3$

Solution: $\log_b a=2$, so case 3 says T(n) is $\Theta(n^3)$.

Master method: Ex 6

Form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example: $T(n) = T(n/2) + 1$ (Binary search)

Solution: $\log_b a=0$, so case 2 says T(n) is $\Theta(\log n)$.

Master method: Ex 7

Form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$, provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example: $$T(n) = 2T(n/2) + \log n$$ (Heap construction)

Solution: $\log_b a=1$, so case 1 says T(n) is Θ(n).

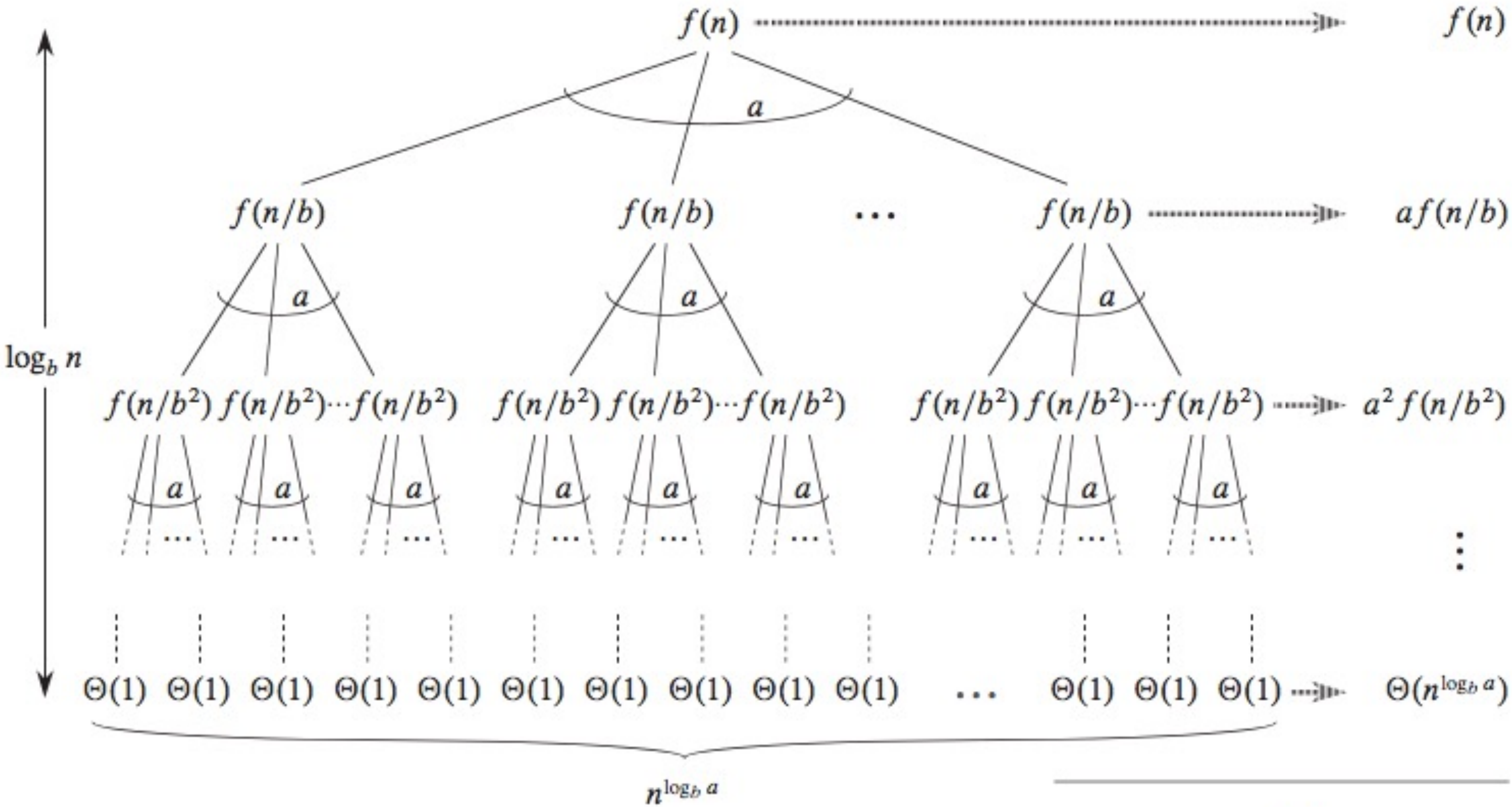
Iterative Justification of the Master Theorem

Use iterative substitution to find a pattern:

$$
\begin{aligned}
T(n) &= aT(n/b) + f(n) \\
&= a(aT(n/b^2)) + f(n/b)) + f(n) \\
&= a^2T(n/b^2) + af(n/b) + f(n) \\
&= a^3T(n/b^3) + a^2f(n/b^2) + af(n/b) + f(n) \\
&= \ldots \\
&= a^{\log_b n}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \\
&= n^{\log_b a}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)
\end{aligned}
$$

We then distinguish the three cases as

- Case 1: The first term is dominant
- Case 2: Each part of the summation is equally dominant
- Case 3: The second term is dominant



Total: $\Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)$

- Case 1: The first term is dominant
- Case 2: Each part of the summation is equally dominant
- Case 3: The second term is dominant

Integer Multiplication

Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

- We can then define I*J by multiplying the parts and adding:

$$\begin{aligned} I * J &= (I_h 2^{n/2} + I_l) * (J_h 2^{n/2} + J_l) \\ &= I_h J_h 2^n + I_h J_l 2^{n/2} + I_l J_h 2^{n/2} + I_l J_l \end{aligned}$$

- So, $T(n) = 4T(n/2) + n$, which implies $T(n)$ is $\Theta(n^2)$.
- But that is no better than the algorithm we learned in grade school.

Improved Integer Multiplication

Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

- Observe that there is a different way to multiply parts:

$$\begin{aligned} I * J &= I_h J_h 2^n + [(I_h - I_l)(J_l - J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + [(I_h J_l - I_l J_l - I_h J_h + I_l J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + (I_h J_l + I_l J_h) 2^{n/2} + I_l J_l \end{aligned}$$

- So, T(n) = 3T(n/2) + n, which implies T(n) is $\Theta(n^{\log_2 3})$.
- Thus, T(n) is $O(n^{1.585})$.