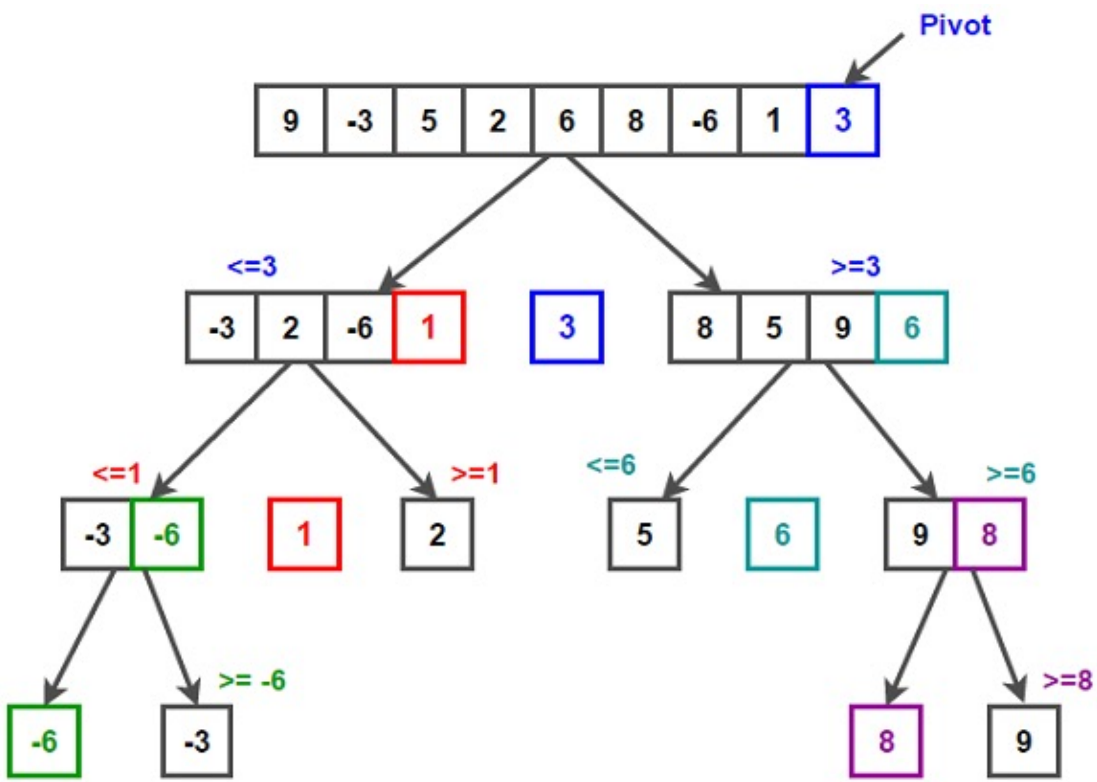
Quicksort

CLRS 7.1 – 7.4
(+ some supplemental material)



Recap

- **Divide-and-conquer** is a general algorithm design paradigm:
  - Divide: divide the input data S into disjoint subsets
  - Conquer: solve the subproblems associated with smaller subproblems
    - the base case for the recursion are subproblems of size 0 or 1
  - Combine: combine the solutions for subproblems into a solution for S

- Merge sort was a divide and conquer approach
  - Divide into 2 lists
  - Recursively sort lists
  - **Merge** two now sorted lists into one sorted list

- Our 2 lists were not sorted *with respect to each other*, so the bottleneck of this approach was in the **merge** step.

- What if we were more careful with how we divided starting array?

Quicksort

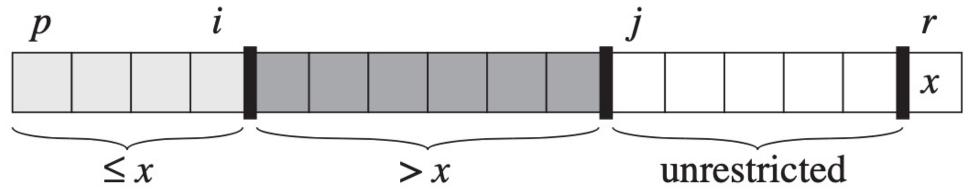
Quicksort works on an input sequence with *n* elements and consists of three steps:

- Divide: **partition** the *n*-element sequence to be sorted into lists based on a select **pivot** element x
  - subsequence 1: list of other elements $\leq x$
  - subsequence 2: list of other elements $> x$
- Conquer: sort the two subsequences recursively using quicksort
- Combine: subsequences are already sorted internally and with respect to each other, so no work is needed to combine


```
QUICKSORT(A, p, r)
if p < r
    q = PARTITION(A, p, r)
    QUICKSORT(A, p, q - 1)
    QUICKSORT(A, q + 1, r)
```

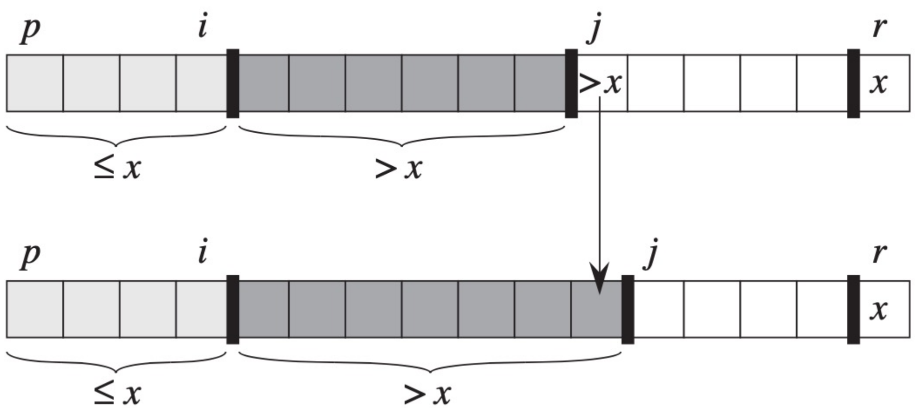
Quicksort is initially called as *Quicksort(A, 1, A.length)*

Quicksort partition

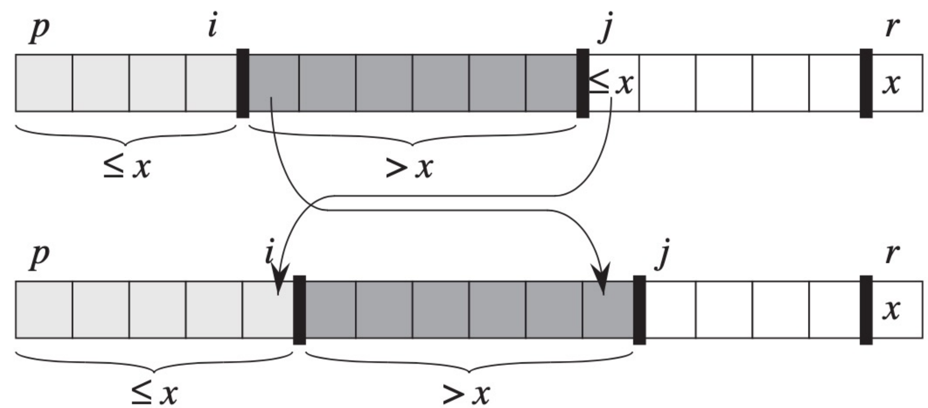


- Maintain four partitions
- As the next item is processed, compare it to the pivot to determine which subsequence it belongs to

If next item is greater than the pivot



If next item is less than or equal to the pivot



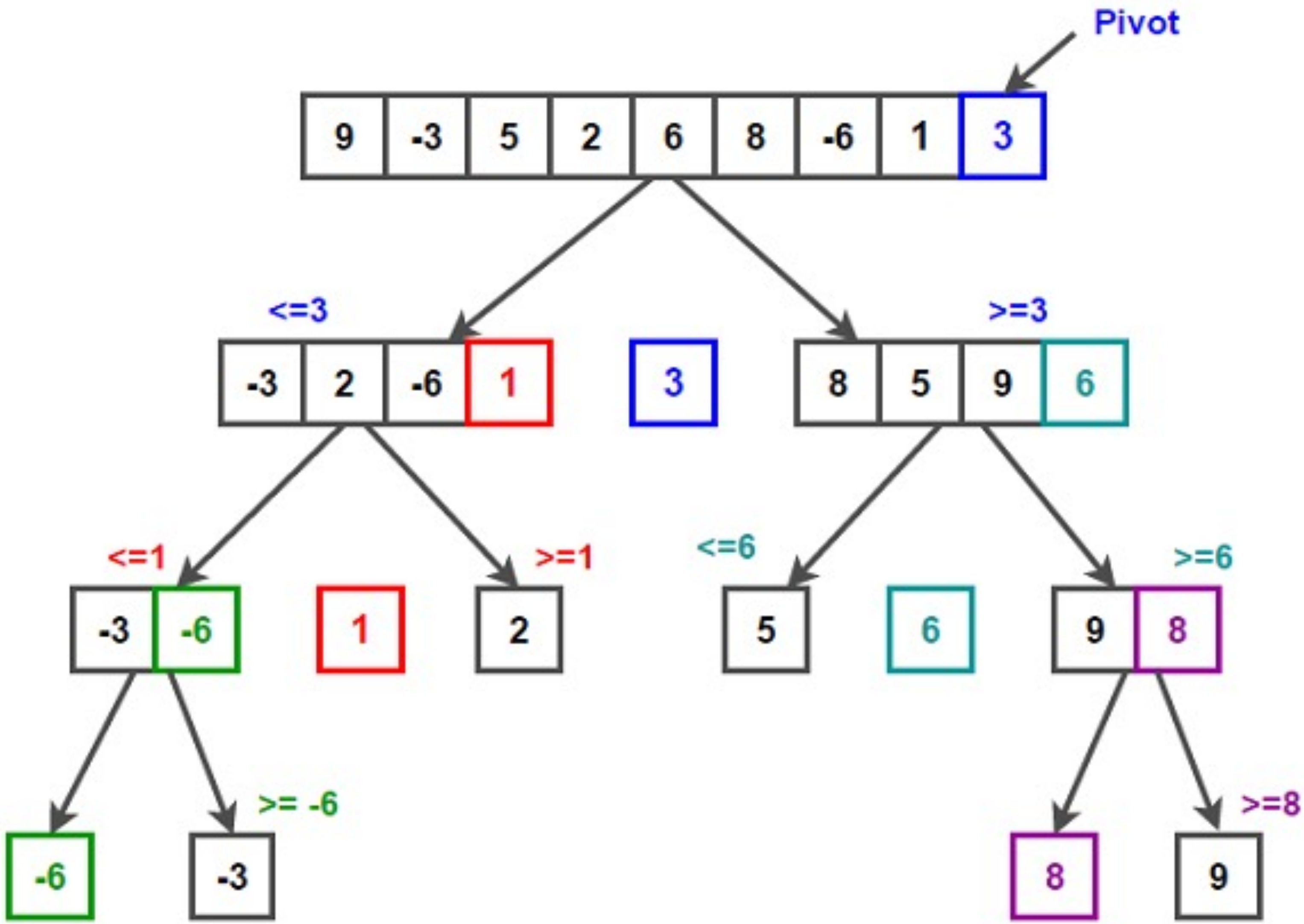
PARTITION(A, p, r)

```
x = A[r]
i = p − 1
for j = p to r − 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i + 1
```

Partitions n elements **in-place** in $O(n)$ time

Illustration of the execution of quicksort

- Using the last element as pivot



Quicksort run time

- Like mergesort, the non-recursive overhead at each level is $O(n)$
  - This is the cost of the partitioning method
- Q: How many recursive calls can be made in the worst case?
  - $O(n)$ – this happens if we pick a **bad pivot** each time -- the minimum or maximum element
  - Example: if the list is already sorted
- Worst-case run time: $O(n^2)$ -- happens if we pick a bad pivot each time
- Best-case run time: $O(n \log n)$ -- happens if we pick a good pivot each time

- We can argue about the average case, provided that we know some information about the input sequence
  - **Assuming the input list is randomly distributed**, we can show that the last element is **usually** a good pivot
  - In this case, the previous version of quicksort runs in $O(n \log n)$ average time

- Rather than making assumptions about the input, we can instead use a randomized version of quicksort which picks a random pivot each time.

Randomized algorithms

There are some variations

- Las Vegas algorithms
  - Correct output guaranteed
  - Randomization makes **good average running time**
  - Introduced by Hungarian professor at University of Chicago László Babai
  - Gambles with running time

- Monte Carlo algorithms
  - Running time guaranteed
  - Randomization makes **probably correct output**
  - Gambles with correctness

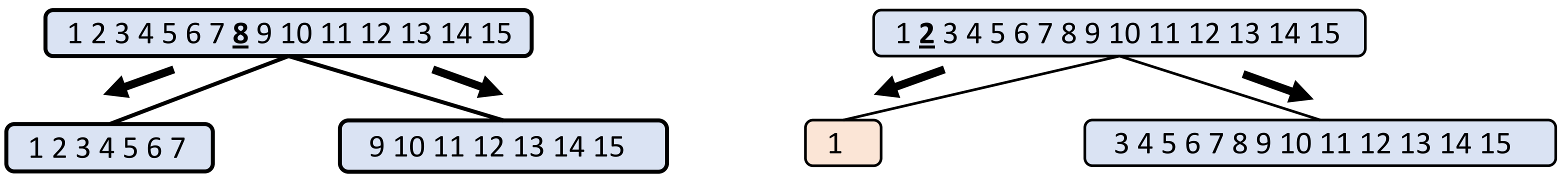
The following randomized version of quicksort is a Las Vegas algorithm.

- The only difference is that we pick always pick a random pivot, rather than the last element as pivot

Expected running time of randomized quicksort

Consider a recursive call of quick-sort on a sequence of size *s*

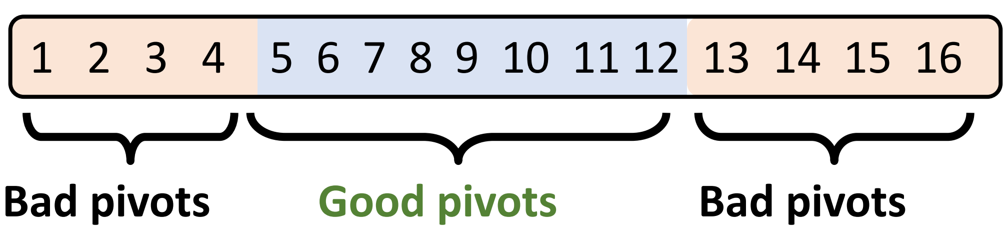
- **Good call:** the sizes of L and G are each less than *3s/4*
- **Bad call:** one of L and G has size greater than *3s/4*



Good call **Bad call**

A call is **good** with probability ½

- 1/2 of the possible pivots cause good calls
- We can show this by visualizing an already sorted list, and counting the number of good pivots



Expected running time of randomized quicksort

Probabilistic Fact: The expected number of coin tosses required in order to get *k* heads is *2k*.

For a node of depth i, we expect

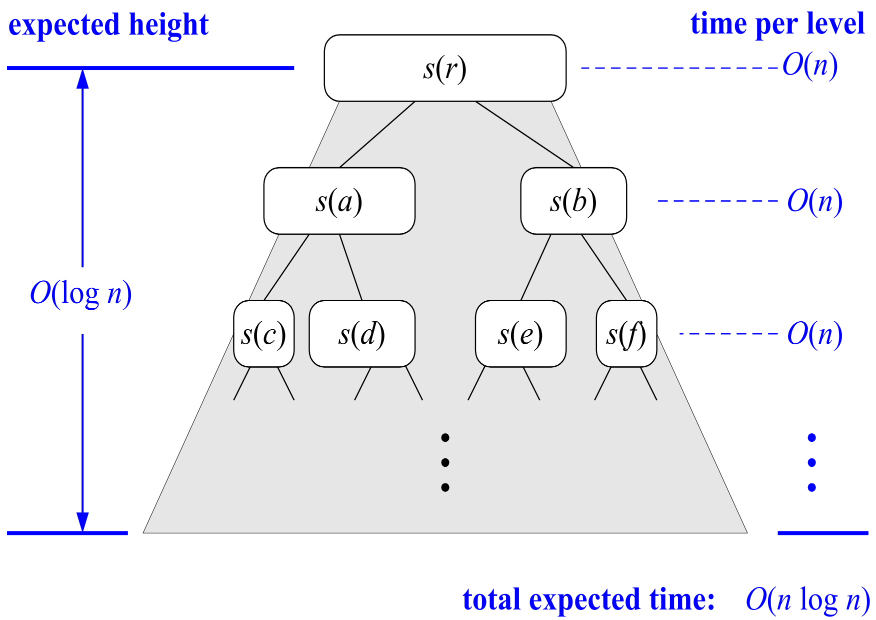
- $i/2$ ancestors are good calls
- size of the input sequence for the current call is at most $\left(\frac{3}{4}\right)^{\frac{i}{2}} n$

For a node of depth $2\log_{4/3} n$ the expected input size is one

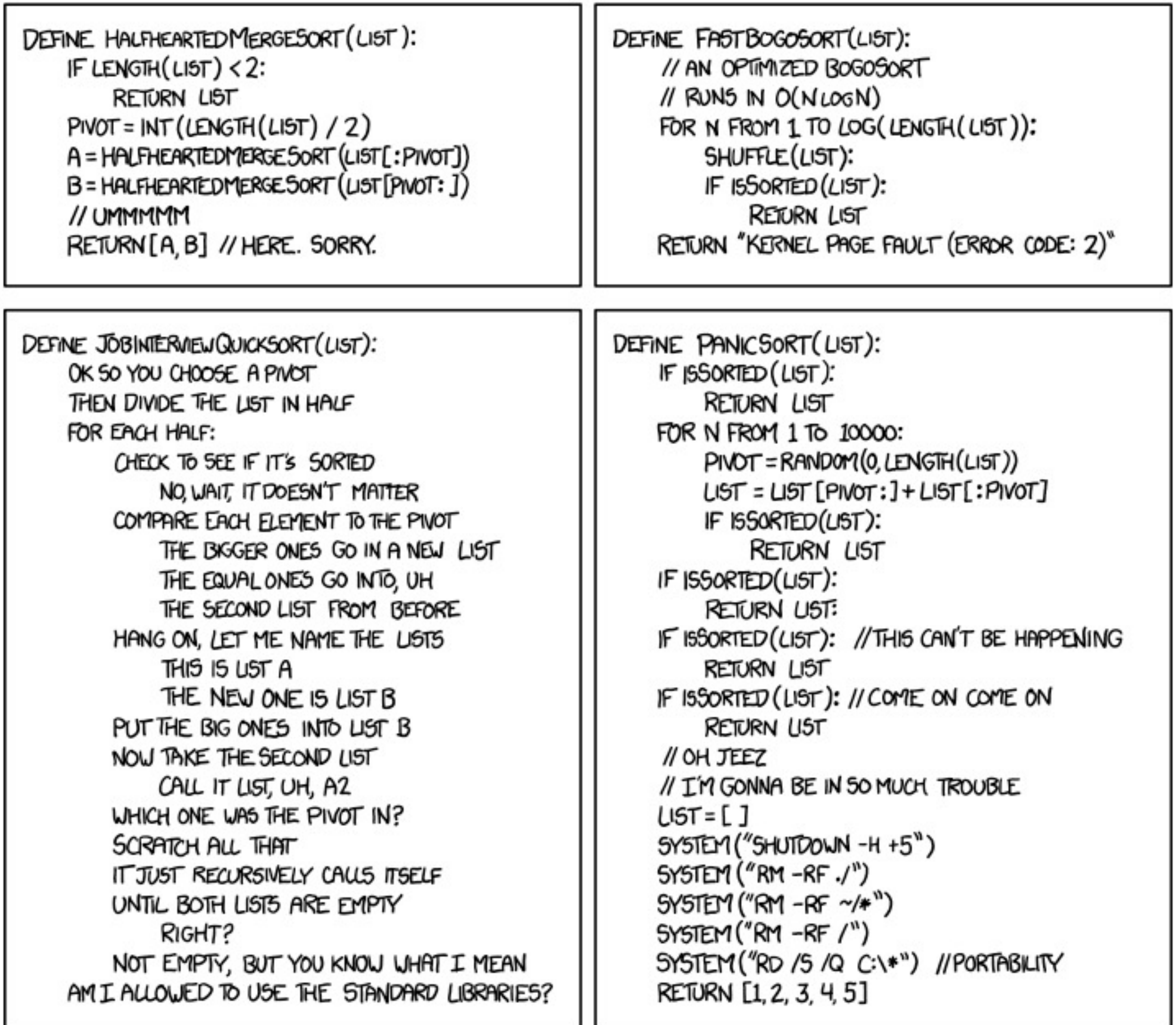
- the expected height of the quick-sort tree is $O(\log n)$

The amount of work done at the nodes of the same depth is $O(n)$

Thus, the expected running time of randomized quick-sort is $O(n \log n)$



xkcd #1185 – ineffective sorts



Other: nuts and bolts

You are given a collection of *n* bolts of different widths, and *n* corresponding nuts.

- You can test whether a given nut and bolt fit together, from which you learn whether the nut is too large, too small, or an exact match for the bolt.
- The differences in size between pairs of nuts or bolts are too small to see by eye, so you cannot compare the sizes of two nuts or two bolts directly.
- You are to match each bolt to each nut.

Give an efficient algorithm to solve the nuts and bolts problem.