Sorting Lower Bounds
Linear sorting

CLRS 8.1 – 8.4
(+ some supplemental material)

Comparison-based sorting

- Recall – Sorting
  - input: A sequence of n values $x_1, x_2, \dots, x_n$
  - output: A permutation $y_1, y_2, \dots, y_n$ such that $y_1 \le y_2 \le \cdots \le y_n$

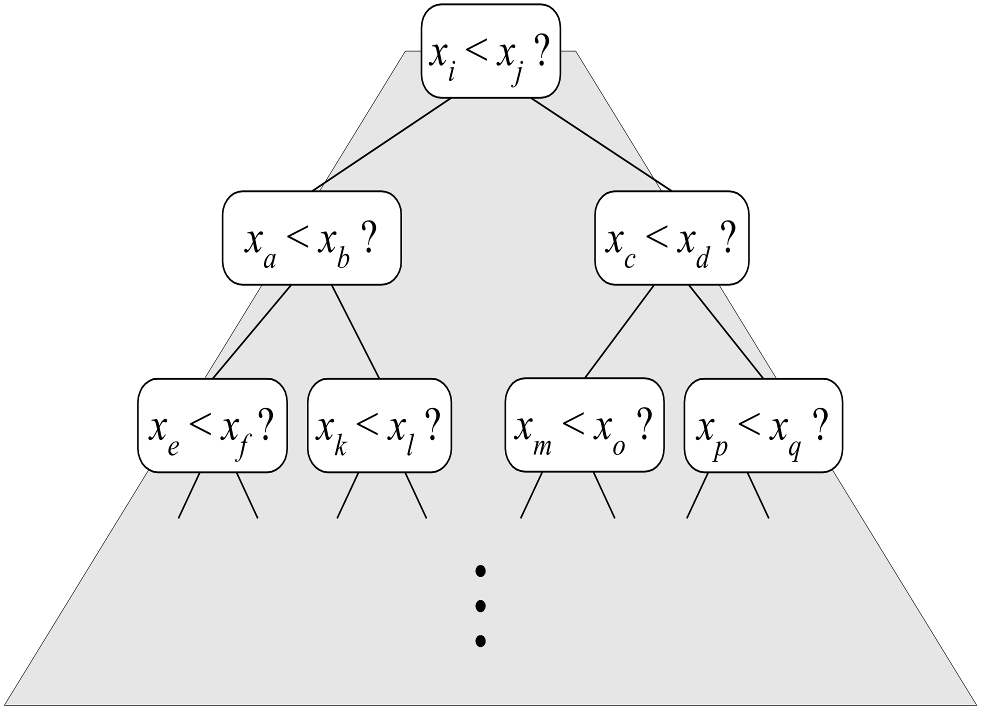
- Many algorithms are comparison based
  - they sort by making comparisons between pairs of objects
  - ex: selection-sort, insertion-sort, heap-sort, merge-sort, quick-sort, …
  - best so far runs in $O(n \log n)$ time… can we do better?

- Let's derive a lower bound on the running time of any algorithm that uses comparisons to sort n elements

Counting comparisons

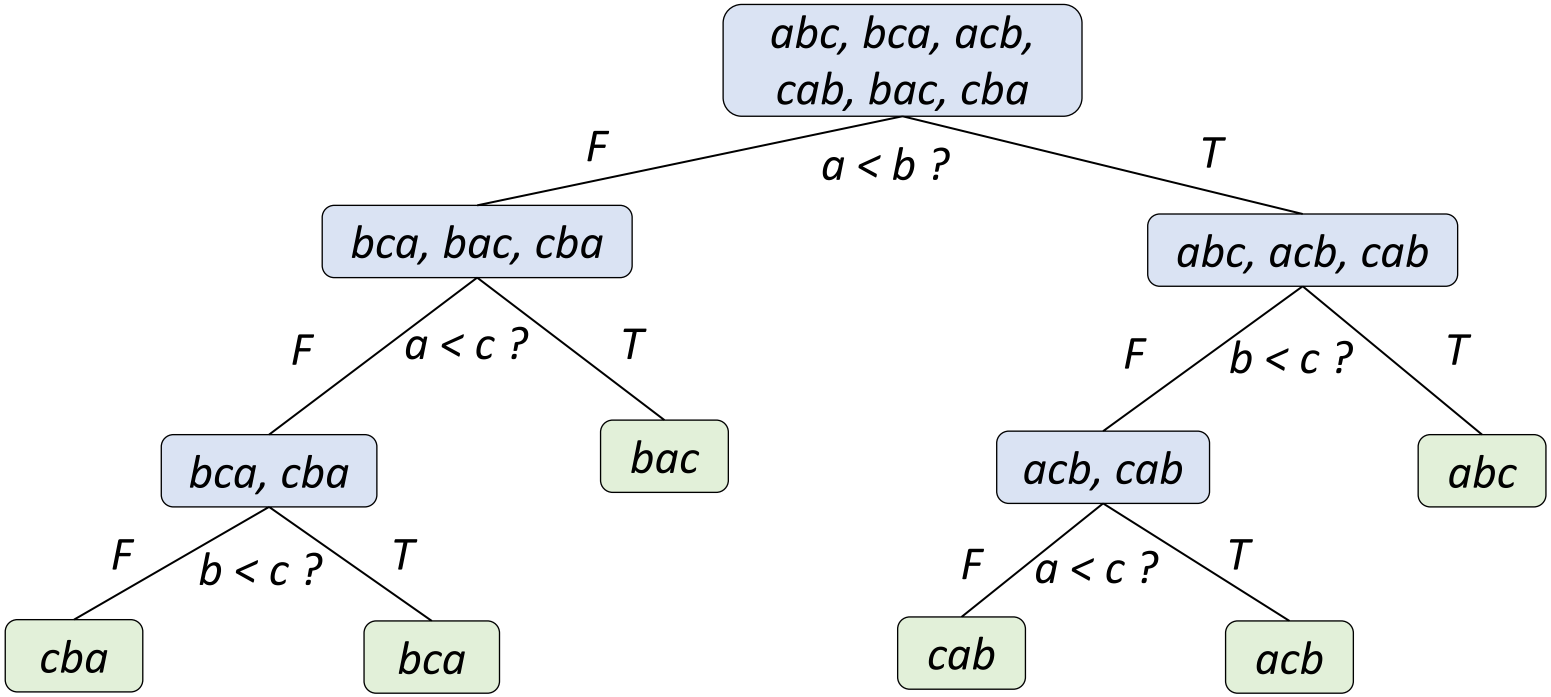
A decision tree represents every sequence of comparisons that an algorithm might make on an input of size n

- **each possible run of the algorithm corresponds to a root-to-leaf path**
- at each internal node a comparison $x_i < x_j$ is performed and branching made
- nodes annotated with the orderings consistent with the comparisons made so far
- leaf contains result of computation (a total order of elements)



Decision Tree Example

- Algorithm: insertion sort
- Instance (n = 3): the numbers a, b, c



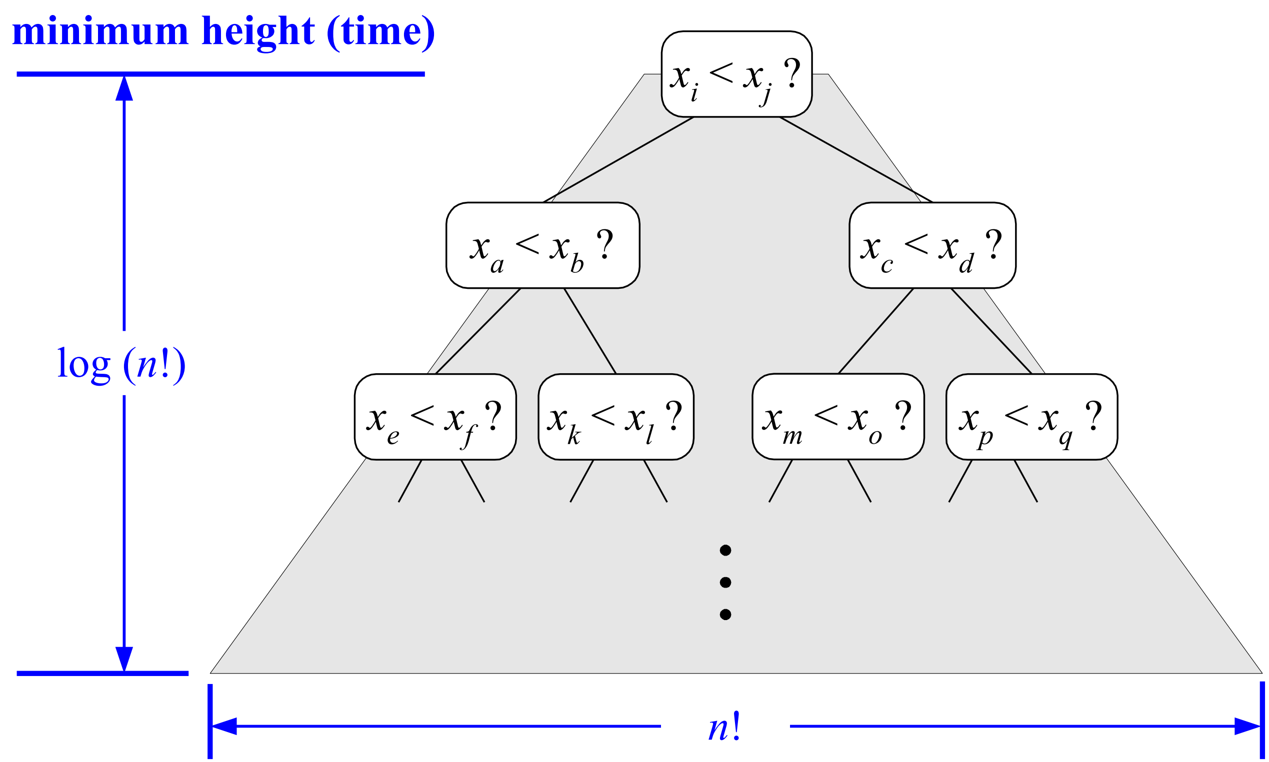
Height of Decision Tree

Theorem: Any decision tree sorting n elements has height $\Omega(n \log n)$.

Proof: There are $n!$ leaves. A binary tree of height h has at most 2^h leaves. So

$$\begin{aligned} 2^h &\geq n! \\ h &\geq \log(n!) \\ &\geq c \cdot \log(n^n) \\ &= c \cdot n \log(n) \end{aligned}$$

Thus, $h \in \Omega(n \log n)$.



Corollary: Any sorting algorithm that uses only comparisons takes $\Omega(n \log n)$ in the worst case.

Linear time sorting

Any comparison-based sorting algorithm runs in $\Omega(n \log n)$ time in the worst case.

To achieve linear-time sorting of *n* elements:

- (!!!) Assume **keys** are **integers** in the **range** $[\mathbf{0}, \boldsymbol{k}]$
- We can use other operations instead of comparisons
- We can sort in linear time when k is small enough
  - Note: we cannot assume this for just any problem with integers !!!

Some sorting algorithms which are **not** comparison-based

- Counting sort
- Radix sort
- Bucket sort

Counting sort

Input: array $A[1 \dots n]$ of integers, each in the range $[0, k]$

<u>Steps:</u>

- Create counting array $C[0 \dots k]$ with values initially zero
- For each of the n items $a \in A$, increment the value of $C[a]$
- For each of the k items in C, add adjacent values
- Now, $C[i]$ contains the number of elements less than or equal to i
- Create a new output array $B[1 \dots n]$ which will hold sorted elements
- For each of the n items $a \in A$ (starting at the end),
  - Set $B\big[C[a]\big] = a$
  - Decrement $C[a]$

COUNTING-SORT(A, B, n, k)

```
let C[0..k] be a new array
for i = 0 to k
    C[i] = 0
for j = 1 to n
    C[A[j]] = C[A[j]] + 1
for i = 1 to k
    C[i] = C[i] + C[i-1]
for j = n downto 1
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] - 1
```

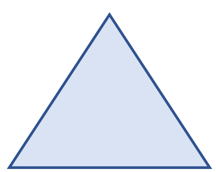
https://algorithm-visualizer.org/divide-and-conquer/counting-sort

Notable properties of counting sort

- **Run time:**
  - $\boldsymbol{O(n+k)}$
  - $\boldsymbol{O(n)}$ **when** $\boldsymbol{k = O(n)}$
  - Ex: if all integers are in the range $[0, 100n]$, then counting sort is $O(n)$
  - Ex: if all integers are in the range $[0, n^5]$, then counting sort is $O(n^5)$
  - Ex: if all integers are in the range $[0, 2^n]$, then counting sort is $O(2^n)$
- It is **stable**: numbers with the same value appear in the output array in the same order as they do in the input array
  - Important when we are **sorting multiple times** based on different attributes
  - Ex: sort a list of names by first name, then sort by last name
- Ex: Unsorted sequence (**B**, **b**, a, c). Suppose B = b and a < b < c.
  - Stable sorted: (a, **B, b**, c)
  - Unstable sorted: (a, **b, B**, c)

In general, we can choose two:

Efficient run time complexity



Stable

Efficient space complexity (in-place)

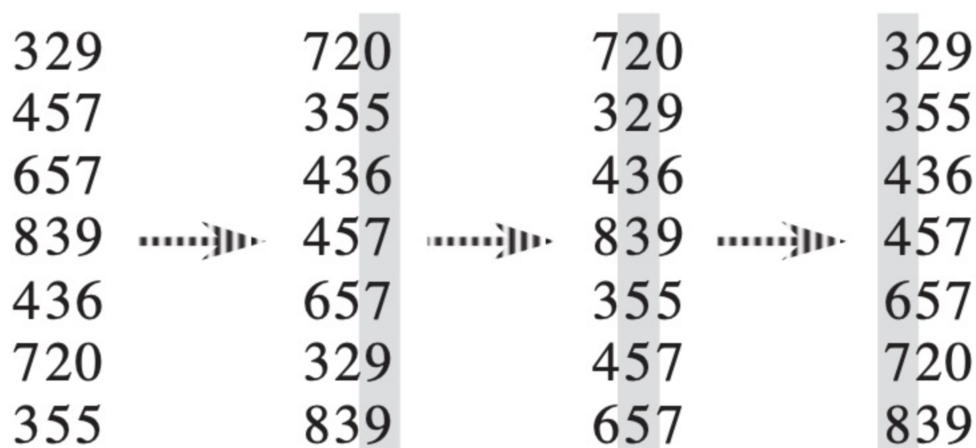
Radix sort

Input: array $A[1 \dots n]$ of n integers where

- each integer is represented as d keys: $\boldsymbol{x_d x_{d-1} \dots x_2 x_1}$
- $\boldsymbol{x_d}$ is the most significant key/dimension; $\boldsymbol{x_1}$ is the least significant key/dimension
- all $\boldsymbol{nd}$ keys are in the range $[\mathbf{0}, \boldsymbol{k}]$

```
RADIX-SORT(A, d)
for i = 1 to d
    use a stable sort to sort array A on digit i
```

Here, we represent an integer key in base 10 (so, all keys are in the range [0,9].
In this case, $d = 3$.



Notable properties of radix sort

- **Run time:**
  - $\boldsymbol{O(d(n+k))}$
  - $O(n)$ when d is constant and $k = O(n)$

- In the last example, we represented integers in base 10.
  - Suppose our maximum integer is N, and $N = O(n^c)$ for a constant c
  - Then the number of keys needed for each integer is $d = \log_{10} N = O(\log n)$
  - Total run time: $O(n \log n)$

- We can do better!! What if we represented integer keys in base n?
  - Suppose our maximum integer is N, and $N = O(n^c)$ for a constant c
  - Then the number of keys needed for each integer is $d = \log_n N = c = O(1)$
  - Total run time: $O(n)$

- Ex: if all integers are in the range $[0, n^3]$, then representing the integers in base n allows radix sort to run in $O(n)$ time