Vector Math for 3D Computer Graphics

An Interactive Tutorial

Second Revision, July 2000

This is a tutorial on *vector* algebra and *matrix* algebra from the viewpoint of computer graphics. It covers most vector and matrix topics needed for college-level computer graphics text books. Most graphics texts cover these subjects in an appendix, but it is often too short. This tutorial covers the same material at greater length, and with many examples.

Although primarily aimed at computer science students, this tutorial is useful to all programmers interested in 3D computer graphics or 3D computer game programming. In spite of their appealing blood-and-gore covers, mass trade books on game programming require the same understanding of vectors and matrices as more staid text books (and usually defer these topics to the same skimpy mathematical appendix).

This tutorial is useful for more than computer graphics. Vectors and matrices are used in all scientific and engineering fields, and any other field that uses computers (are there any that don't?) In many fields, the vocabulary used for vectors and matrices does not match that used in computer graphics. But the ideas are the same, and reading these notes will take only a slight mental adjustment.

These notes assume that you have studied plane geometry and trigonometry sometime in the past. Notions such as "point", "line", "plane", and "angle" should be familiar to you. Other notions such as "sine", "cosine", "determinant", "real number", and the common trig identities should at least be a distant memory.

These pages were designed at 800 by 600 resolution with "web safe" colors. They have been (somewhat) tested with not-too-old versions of Netscape Navigator and Internet Explorer, using "Times Roman" font (the usual browser default font). Many pages require Javascript, and some pages require Java. If you lack these (or are behind a firewall that does not allow these inside) you will be able to read most pages, but the interactive features will be lost.

Some sections are more than three years old and have been used in several lecture sections (and hence are "classroom tested" and likely to be technically correct and readable). Other sections have just been written and might fall short of both goals.

- Return to the list of courses.
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created 08/19/97; revised 08/01/00; bug (ant) fixes 09/14/00

CHAPTER 1 --- Vectors, Points, and Column Matrices

This chapter discusses the objects of computer graphics---vectors and points---and how they are represented in a computer---as column matrices. A column matrix is a mathematical object that has many uses besides its use in computer graphics. These notes discuss only the aspects that are used in computer graphics.

Chapter Topics:

- *Computer graphics as modeling and viewing.*
- *Geometrical points and vectors.*
- *Column and row matrices.*
- *Calculating displacements.*
- *Equality of column matrices.*
- *Names for column matrices.*
- *Representing points with column matrices.*

Computer graphics books use one of two ways to represent points and vectors. Some books use row matrices; other books use column matrices. The two methods are exactly equivalent (although some formulae have to be adjusted). These notes use column matrices.

Some graphics books use the term "column vector" for the object that these notes call a "column matrix." This is just a variation in terminology and does not affect the concepts or formulae presented here.

QUESTION 1:

(Review:) What two types of objects are represented with column matrices?

Click Here after you have answered the question

A good answer might be:

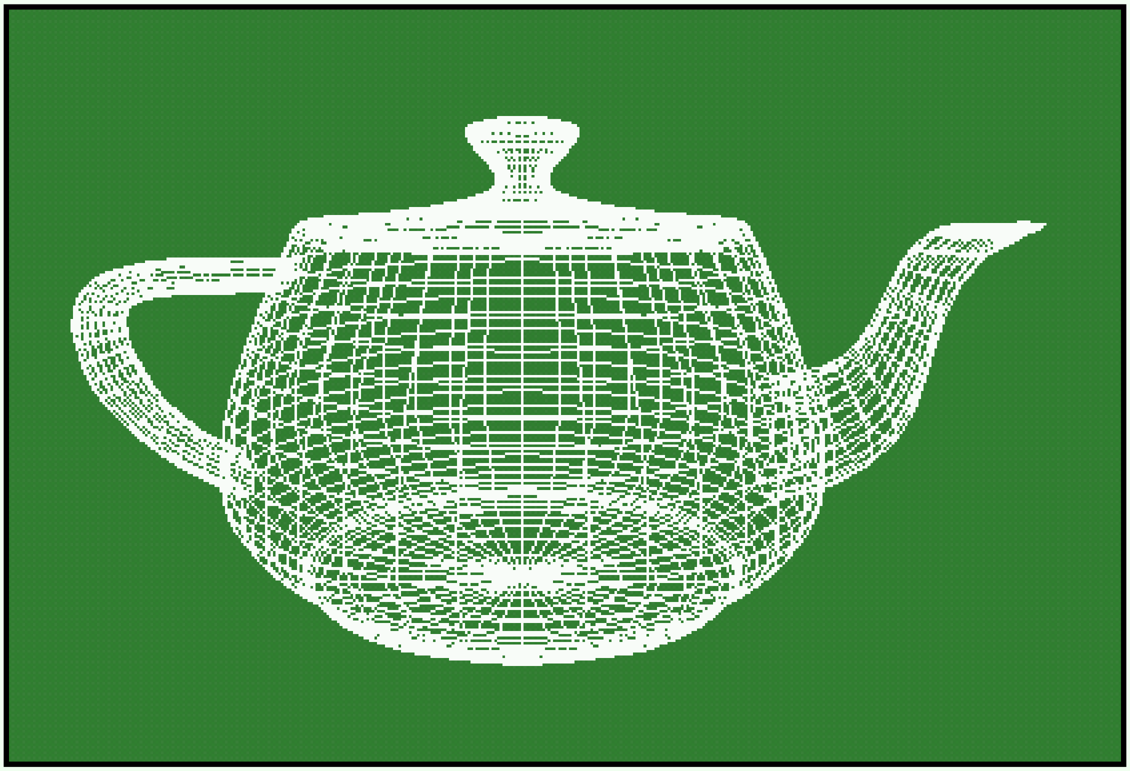
(1) Points, and (2) Vectors.

Virtual Tourist

By "geometrical point" these notes will mean the "point" of plane geometry and solid geometry. Hopefully you have studied geometry sometime in the past.

Computer graphics consists of two activities: (1) Creating an imaginary world inside a computer, and (2) producing two dimensional images of that world from various viewpoints. A graphics program is like a tourist wandering through a fantastic landscape taking pictures with a camera. With computer animation, the virtual tourist is equipped with a movie camera.

The imaginary landscape is built of objects in three dimensional space. Each object is a set of points and connections between points that form the edges of a 3D solid. Here, for example is a set of points that model a teapot (from the Open GL Utility Toolkit):



It may be hard to see, but the figure consists of points and the line segments that connect them. To make a realistic picture of a teapot many operations are needed to fill in the area between line segments, apply texture and lighting models, and other rendering operations. But the fundamental object is modeled with connected points.

QUESTION 2:

What is a line segment?

Click Here after you have answered the question

What is a line segment?

A good answer might be:

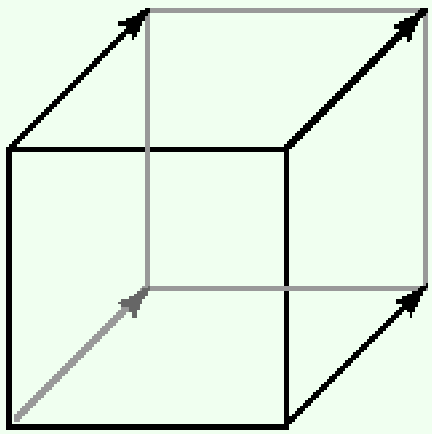
A hard question. Again, you should have an idea of what a line is from your study of geometry.

Vector

In geometry, a **point** is a location in space. A point does not have any size, its only property is location. In computer graphics, points will mostly be the vertices of 3D figures.

A **vector** has two properties: length and direction. A vector does not have a fixed location in space. The idea is that points are used to encode location, and vectors are used to encode connections between points. This may seem rather odd, but it makes 3D computer graphics easier.

The combination of "distance and direction" is sometimes called a **displacement**. Often the same displacement (i.e. just one displacement) is applied to each of several points. For example, consider this cube:



The front face contains four vertices (four points). If you move the same distance and direction from each of these points you reach the four vertices of the back face.

The "same distance and direction" is a vector, shown here as a line with an arrow head. The diagram shows this one vector four times, once for each point of the face.

QUESTION 3:

Say that it is high noon on August 24th and you are at the beach and the Sun shining down. Is the light from the Sun shining down in the same direction for everyone on the beach?

Click Here after you have answered the question

A good answer might be:

Yes. The "direction to the Sun" is the same for all locations on the beach. In other words, it is a vector.

Column Matrix

Sometimes (as in the question) we are interested only in direction. A vector is used for this, but its length does not matter. Often it will be given a length of one.

A geometric vector may be represented with a list of numbers called a **column matrix**. A column matrix is an ordered list of numbers written in a column. Here is an example of a column matrix:

$$\begin{bmatrix} 2.9 \\ -4.6 \\ 0.0 \end{bmatrix}$$

Each number of the column matrix is called an **element**. The numbers are *real numbers*. The number of elements in a vector is called its **dimension**. A **row matrix** is an ordered list of numbers written in a row. Here is an example of a row matrix:

```
(12.5, -9.34)
```

To be consistent, our vectors will always be represented with column matrices. Some books represent vectors with row matrices, which makes no fundamental difference.

QUESTION 4:

How many elements are in each column matrix?

Column Matrix	$\begin{bmatrix} 2.9 \\ -4.6 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -13.45 \\ 92.4 \\ 2.0 \\ -42.01 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \end{bmatrix}$

Dimension	. ? .	. ? .	. ? .

Click Here after you have answered the question

A good answer might be:

Column Matrix	$\begin{bmatrix} 2.9 \\ -4.6 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} -13.45 \\ 92.4 \\ 2.0 \\ -42.01 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 0 \\ 5 \\ 1 \end{bmatrix}$
Dimension	3	4	4

Variables as Elements

The elements of a column matrix can be *variables*:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

The first element in a column matrix is sometimes given the index "0", and sometimes "1".

QUESTION 5:

Is the column matrix $\begin{bmatrix} 2.9 \\ -4.6 \\ 0.0 \end{bmatrix}$ the same as the column matrix $\begin{bmatrix} 2.9 \\ 0.0 \\ -4.6 \end{bmatrix}$?

[Click Here after you have answered the question]

Is the column matrix $\begin{bmatrix} 2.9 \\ -4.6 \\ 0.0 \end{bmatrix}$ the same as the column matrix $\begin{bmatrix} 2.9 \\ 0.0 \\ -4.6 \end{bmatrix}$?

A good answer might be:

No. A column matrix is an ordered list of numbers. This means that each position of the column matrix contains a particular number (or variable.)

Convenient Printing of Column Matrices

Column matrices are awkward in printed text. It is common to print a column matrix like this:

$$(2.9,\ -4.6,\ 0.0)^T$$

The "T" stands for **transpose**, which means to change rows into columns (later on a more elaborate definition of "transpose" will be needed.)

QUESTION 6:

Is (1.2, -3.9, 0.0) equal to $(1.2, -3.9, 0.0)^T$

Click Here after you have answered the question

Is (1.2, -3.9, 0.0) equal to (1.2, -3.9, 0.0)^T

A good answer might be:

No. Column matrices and row column matrices are different types of objects, and cannot be equal.

Column Matrix Equality

Your previous courses may not have distinguished between row matrices and column matrices. They may not have made it clear that *geometric vectors* are different from the *column matrices* that represent them. These differences may seem picky at the moment, but keeping them straight will help you get through the difficult material to come.

Here is what it takes for two row or two column matrices to be equal:

1. Both matrices must be column matrices, or both must be row matrices.
2. Both must have the same dimension (number of elements).
3. Corresponding elements of the matrices must be equal

Only matrices of the same "data type" can be compared. You can compare two three-dimensional column matrices, or two four-dimensional row matrices, and so on. It makes no sense to compare a three-dimensional row column matrix to a three-dimensional column matrix. For example:

(6, 8, 12, -3)^T = (6, 8, 12, -3)^T

(6, 8, 12, -3) = (6, 8, 12, -3)

(6, 8, 12, -3) =/= (-2.3, 8, 12, -3)

(6, 8, 12, -3)^T =/= (6, 8, 12, -3)

(6, 8, 12, -3)^T =/= (6, 8, 12)^T

Sometimes the rules are relaxed and one gets a little sloppy about the distinction between row column matrices and column matrices. But keeping the distinction clear is like "strong data typing" in programming languages. It often keeps you out of trouble.

QUESTION 7:

Are the following column matrices equal?

$(1.53, -0.03, 9.03, 0.0, +8.64)^T$
$(1.53, -0.03, 9.03, 1.0, -8.64)^T$

Click Here after you have answered the question

A good answer might be:

No.

Matrices as Data Types

Here is a scheme for matrix data types. To create a matrix type, fill each blank with one of the choices.

```
        .
        .
        .
        4
        3
        2                row
        1                column
     ______ dimensional _______ matrix
```

For example "3 dimensional column matrix", or "4 dimensional row matrix". In a strongly typed programming language, you might declare a type for each of the matrix types. For example, in C:

```
typedef double colMatrix2[2];
typedef double colMatrix3[3];
...

typedef double rowMatrix2[2];
typedef double rowMatrix3[3];
...
```

Then you would write an equality-testing function for each type:

```
int equalCM2( colMatrix2 x, colMatrix2 y )
{
  return x[0] == y[0] && x[1] == y[1] ;
}

int equalCM3( colMatrix3 x, colMatrix3 y )
{
  return x[0] == y[0] && x[1] == y[1] && x[2] == y[2];
}
```

. . .

This gets awfully tedious, so usually you create a more general data type and hope that the user will keep row matrices and column matrices straight without help from the compiler. An elegant solution is to use object-oriented programming.

Since for us vectors will always be represented with column matrices, we will use only a few types.

QUESTION 8:

Are the following two column matrices equal?

$(2, -1)^T$
$(2.0, -1.0)^T$

Click Here after you have answered the question

$(2, -1)^T$
$(2.0, -1.0)^T$

A good answer might be:

Yes---for us, matrix elements are always real numbers (never integers) so for us "2" is short for "2.0" .

Names for Matrices

It is useful to have names for matrices. Usually in print a column matrix name is a bold, lower case letter. For example:

$\mathbf{a}$ == (1.2, -3.6)
$\mathbf{x}$ == (x_1, x_2, x_3, x_4)
$\mathbf{r}$ == (r_0, r_1)^T

It is conventional to use names from the start of the alphabet for column matrices whose elements we know (like **a** above), and to use names from the end of the alphabet for column matrices whose elements are variables.

Often the names of column matrix elements are subscripted versions of the name of the whole column matrix (like column matrix **r** and its elements r_0 and r_1).

Bold face is hard to do with pencil or chalk, so instead an arrow or a bar is placed over the name:

```
_    ->
x    x
```

QUESTION 9:

Say that you know:

```
x = ( x1, x2 )
y =  ( 3.2, -8.6 )
x =  y
```

$$\mathbf{x} = (x_1, x_2)$$
$$\mathbf{y} = (3.2, -8.6)$$
$$\mathbf{x} = \mathbf{y}$$

What must be true about x_1 and x_2?

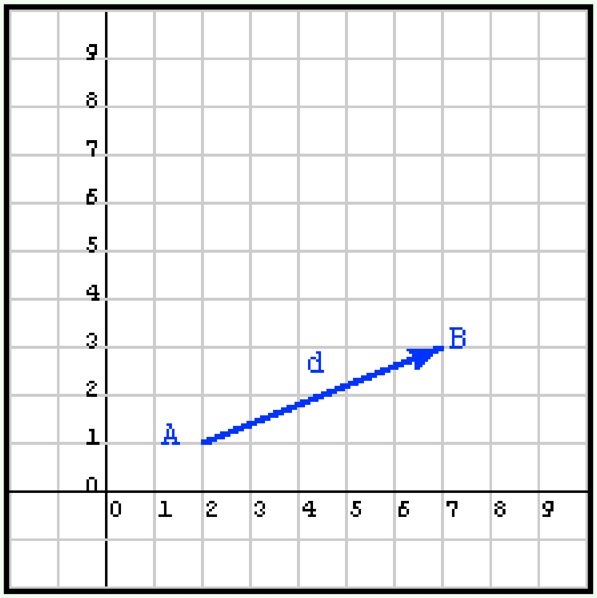
Click Here after you have answered the question

```
x =  ( x1, x2 )
y =  ( 3.2, -8.6 )
x = y
```

A good answer might be:

x_1 = 3.2, and x_2 = -8.6

Column Matrices representing Vectors



Column matrices are used to represent vectors and also used to represent points. In two dimensions, the same data type (two dimensional column matrices) is used to represent two different types of geometrical objects (points and vectors). This is awkward. Later on, this situation will be corrected.

The figure shows a displacement vector representing the difference between two points in the x-y plane. (For now, the examples are in two dimensional space; three dimensional space will come later).

- point A: x=2 y=1. As a column matrix: $(2, 1)^T$
- point B: x=7 y=3. As a column matrix: $(7, 3)^T$

The displacement from A to B can be computed as two separate problems.

- The x displacement is the difference in the X values: 7-2 = 5
- The y displacement is the difference in the Y values: 3-1 = 2

The displacement vector expressed as a matrix is:

$$\mathbf{d} = (5,\ 2)^T$$

Displacement vectors are often visualized as an arrow connecting two points. In the diagram point A is the tail of the vector and point B is the tip of the vector.

QUESTION 10:

The column matrix **d** represents the displacement vector from point A to point B. What column matrix represents displacement from point B to point A?

Click Here after you have answered the question

- point A: x=2 y=1
- point B: x=7 y=3

A good answer might be:

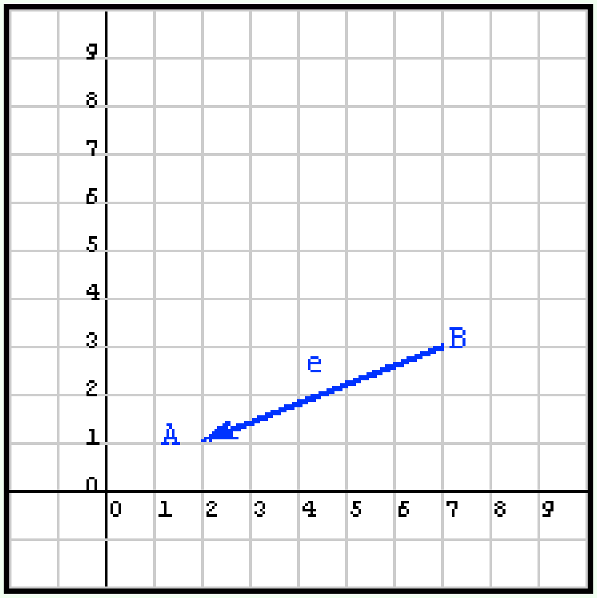
The displacement from point B to point A is:

- The x displacement: 2-7 = -5
- The y displacement: 1-3 = -2

So the column matrix representing the displacement is:

```
e = (-5, -2)^T
```

Displacement



When the points are visited in the opposite order, the displacement vector points in the opposite direction. In the column matrix each element is -1 times the old value.

The displacement from A to B is different from the displacement from B to A. Think of displacement as "directions on how to walk from one point to another." So, if you are standing on point A and wish to get to point B, the displacement $(3, 1)^T$ says "walk 3 units in the positive X direction, then walk 1 unit in the positive Y direction."

Of course, to get from point B to point A you need different directions: the displacement $(-3, -1)^T$ says "walk 3 units in the *negative* X direction, then walk 1 unit in the *negative* Y direction," which puts you back on point A.

The displacement from point **Start** to point **Finish** ==

$$(\text{Finish x} - \text{Start x} , \text{Finish y} - \text{Start y})^T$$

QUESTION 11:

Say that point C is x=4, y=2 and that point D is x=3, y= 5 . What column matrix represents the displacement from C to D?

Click Here after you have answered the question

Say that point C is x=4, y=2 and that point D is x=3, y= 5 . What column matrix represents the displacement from C to D?

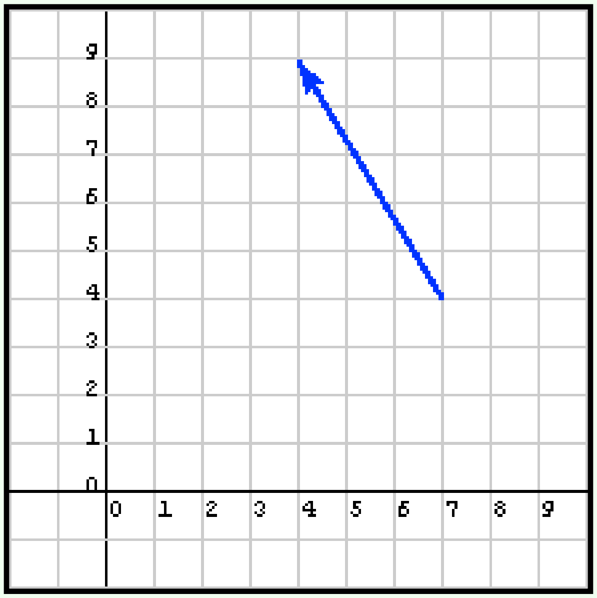
A good answer might be:

- Finish X - Start X = 3 - 4 = -1
- Finish Y - Start Y = 5 - 2 = 3

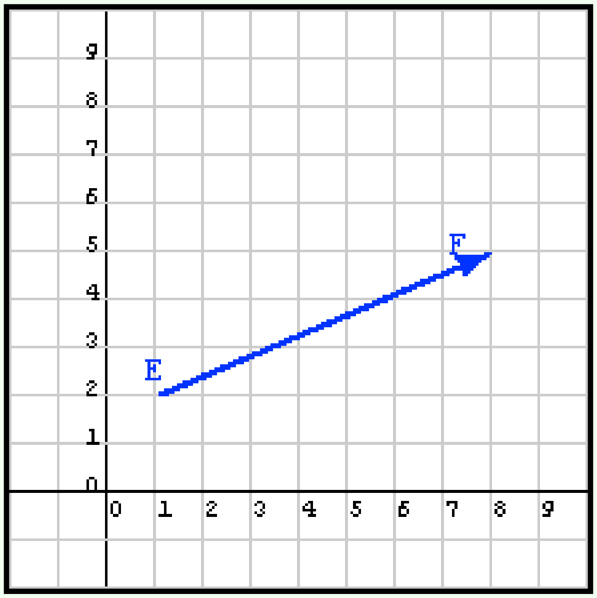
So the column matrix is $(-1, 3)^T$

Reading Displacements from Graph Paper

The first diagram shows a vector between two points. The numerical values can be read off of the graph by counting the number of squares from the tail of the vector to the tip: $(-3, 5)^T$.



First Diagram



Second Diagram

QUESTION 12:

Do that with the second diagram. What is the displacement from E to F?

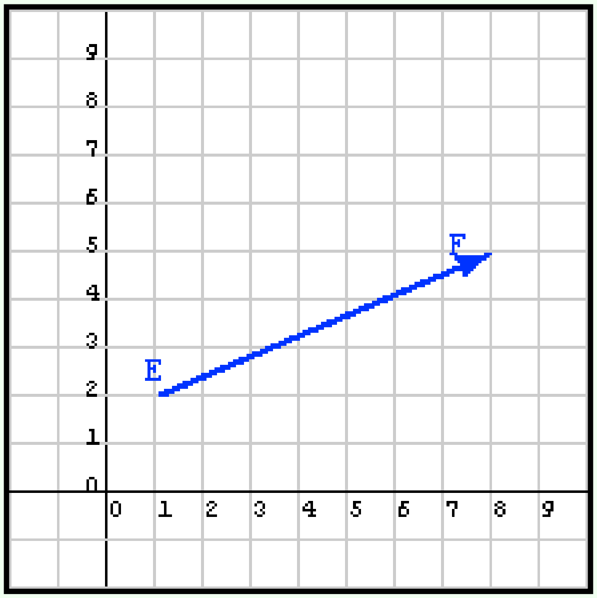
Click Here after you have answered the question

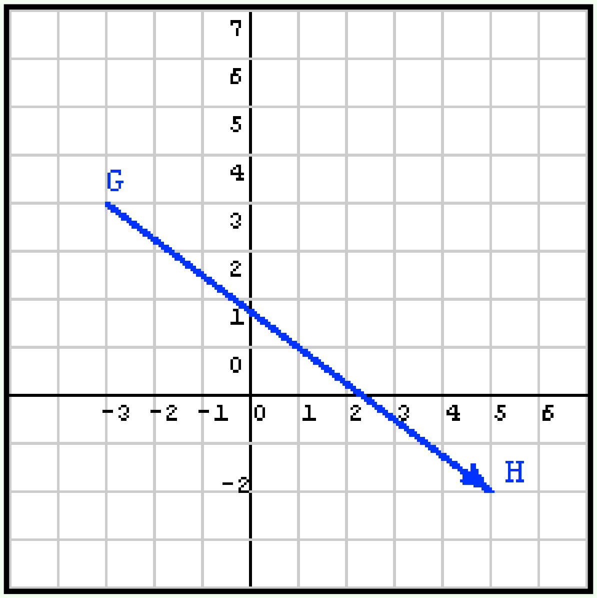
A good answer might be:

$(7, 3)^T$

Checking an Answer

To check your answer, start at the beginning point and follow the directions: walk 7 in X, then walk 3 in Y. If your answer is correct, you end up at the ending point.





QUESTION 13:

What is the displacement from point G, $(-3, 4)^T$ to point H, $(5, -2)^T$?

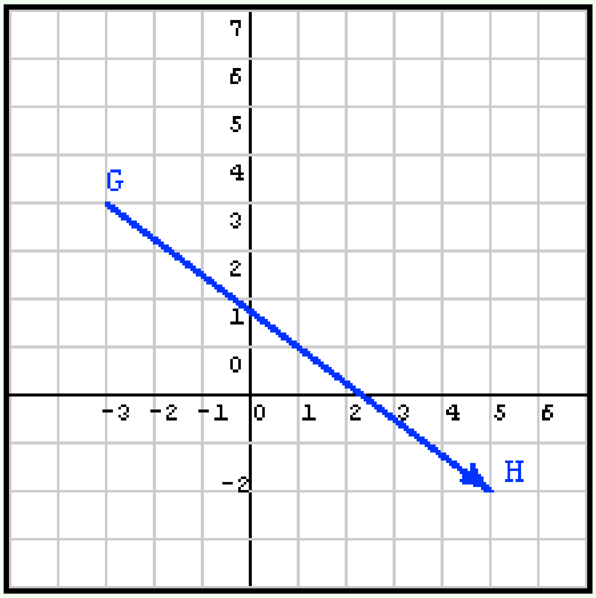
Click Here after you have answered the question

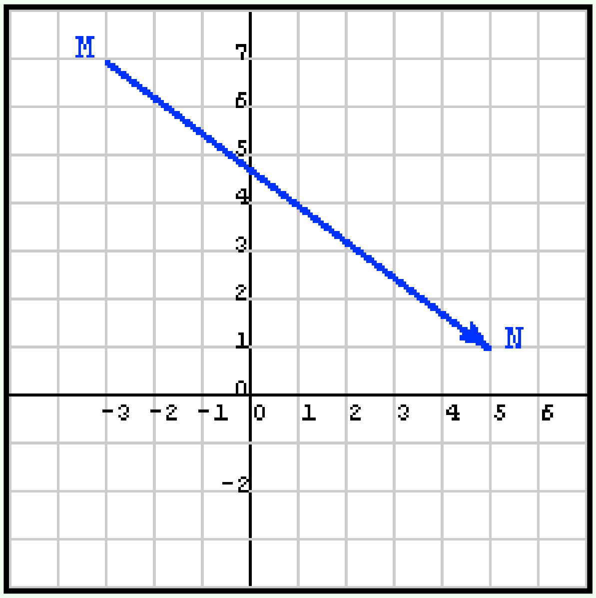
What is the displacement from point G, $(-3, 4)^T$ to point H, $(5, -2)^T$?

A good answer might be:

Subtracting values of the start G from corresponding values of the finish H gives: $(8, -6)^T$

Vectors Don't have a Location





Now calculate the displacement from M to N in the second diagram: Subtracting values of the start N from corresponding values of the finish M gives: $(8, -6)^T$. This is the same as for the first diagram.

Geometrically, the displacement vector from G to H is the same as the displacement vector from M to N. Using the rule for column matrix equality (to review it, click here) the two column matrices are equal. This makes sense because in walking from point G to H you go the same distance and direction as in walking from M to N. The diagrams show the displacements with the same length and direction (but different starting points):

Vectors have no location. In a diagram it is common to draw a vector as an arrow with its tail on one point and its tip on another point. But any arrow with the same length and direction represents the vector.

QUESTION 14:

Is the displacement between two given points unique?

Click Here after you have answered the question

Is the displacement between two given points unique?

A good answer might be:

Yes.

Subtracting Points

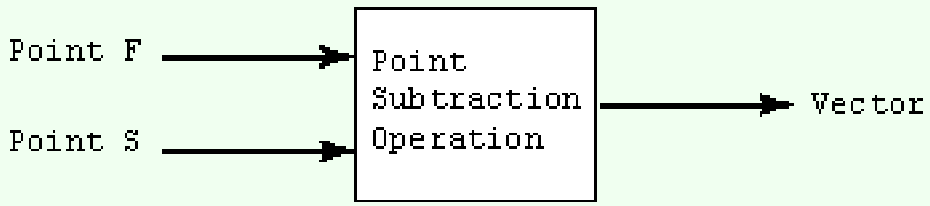
No, I'm not talking about your midterm

The previous formula for calculating a displacement vector (click here to review it) can be written as:

Displacement from point S (start) to point F (finish):

$$F - S = (Xf , Yf)^T - (Xs , Ys)^T = (Xf-Xs , Yf-Ys)^T$$

In this operation two points are used to produce one vector:



Don't skip over this observation with a yawn. Think a few careful thoughts now to avoid confusion later on. It is slightly odd that a "minus" with two objects of one type produces an object of another type.

QUESTION 15:

(Thought question:) If one three dimensional point is subtracted from another, is the result a vector?

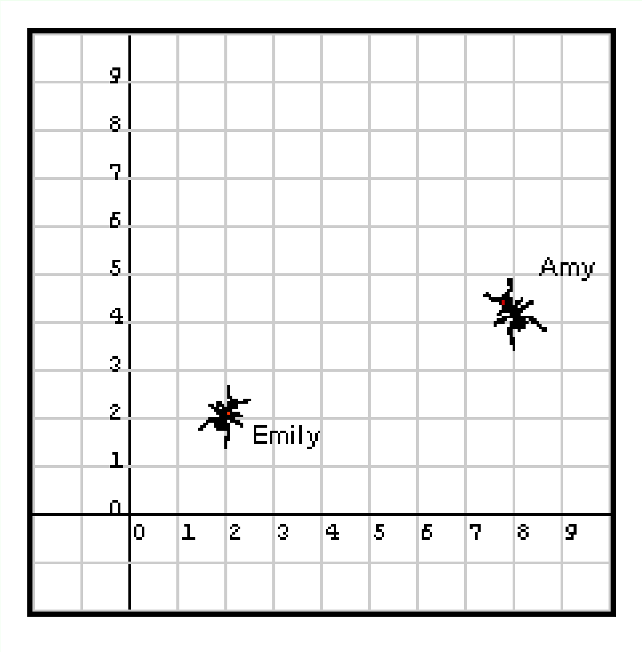
Click Here after you have answered the question

(Thought question:) If one three dimensional point is subtracted from another, is the result a vector?

A good answer might be:

Yes. These ideas work in 3D as well as 2D.

Practice with Displacements



It would be a pity to be talking about math without a story problem: Amy the ant at point $(8, 4)^T$ is lost. Find a displacement that will bring her to her friend Emily at $(2, 2)^T$.

QUESTION 16:

Through what displacement should Amy move to find her friend Emily?

Click Here after you have answered the question

A good answer might be:

By subtracting points:

```
     Emily - Amy = ( 2, 2)ᵀ - ( 8, 4)ᵀ = ( -6, -2)ᵀ.
```

Of course by counting squares of the graph paper you will get the same answer.

Real Numbers

With the story problem, there is a temptation to avoid negative numbers and to incorrectly compute the displacement. But it is perfectly OK to have negative displacements; the "negative" part just shows the direction:

Another thing to keep in mind is that the elements of a column matrix are real numbers. Examples often use integers, but that is just to make things easy. There is nothing wrong with the column matrix $(-1.2304, 9.3382)^T$.

QUESTION 17:

- point B = $(4.75, 6.23)^T$
- point A = $(1.25, 4.03)^T$

What is the displacement from point A to point B?

Click Here after you have answered the question

- point B = $(4.75, 6.23)^T$
- point A = $(1.25, 4.03)^T$

A good answer might be:

The displacement is $(4.75, 6.23)^T - (1.25, 4.03)^T = (3.50, 2.20)^T$

You have reached the end of this chapter. Before you go vectoring off to the campus nightspot, perhaps you would like to review some terms:

- The nature of computer graphics
- Geometric points
- Vectors
- Displacement.
- Column matrices.
- Transpose of a column matrix.
- Equality of column matrices.
- Names for column matrices and their elements.
- Representing a point as a column matrix.
- Calculating displacement.

The next chapter will discuss operations on vectors and the equivalent operations with column matrices.

Click here to go back to the main menu.

You have reached the End.

created 08/19/97; revised 08/06/00; corrected 09/14/00

CHAPTER 2 -- Column and Row Matrix Addition

Here are some of the terms we have been using:

- **point**---a geometric object; a location in 3D (or 2D) space.
- **vector**---a geometric object that has properties of direction and length, but no location.
- **column matrix**---an ordered list of numbers arranged into a column.
- **row matrix**---an ordered list of numbers arranged into a row.
- **element**---one of the numbers that makes up a column or row matrix.
- **dimension**---the number of elements in a column or row matrix.
- **displacement**---the difference between two locations, expressed as a vector.

This chapter discusses addition and subtraction of column and row matrices.

QUESTION 1:

(Review question.) What is it called when a row matrix is "flipped" into a column matrix (or when a column matrix is flipped into a row matrix)?

Click Here after you have answered the question

What is it called when a row matrix is "flipped" into a column matrix?

A good answer might be:

Transposition.

Transposition

Your browser must be Java enabled to see this.

Even if a row matrix and a column matrix are the same dimension and contain the same elements, they are considered different types. Remember that a "T" superscript is used when a column matrix is written as a row of numbers. The first element of a column matrix is the topmost (corresponding to the leftmost element when written as a row.)

The picture to the left shows two different ways of writing out the *same column matrix*. To change an element, enter a new value into its box and hit enter. When the column matrix is written in a row (to save space) the superscript "T" shows that it is really a column matrix. Notice that the elements of the column matrix have the same subscripts no matter how the column matrix is displayed.

A "T" superscript on column matrix with means to flip the column into a row resulting in a row matrix.

QUESTION 2:

What is $(1, 2, 3)^{T\,T}$

Click Here after you have answered the question

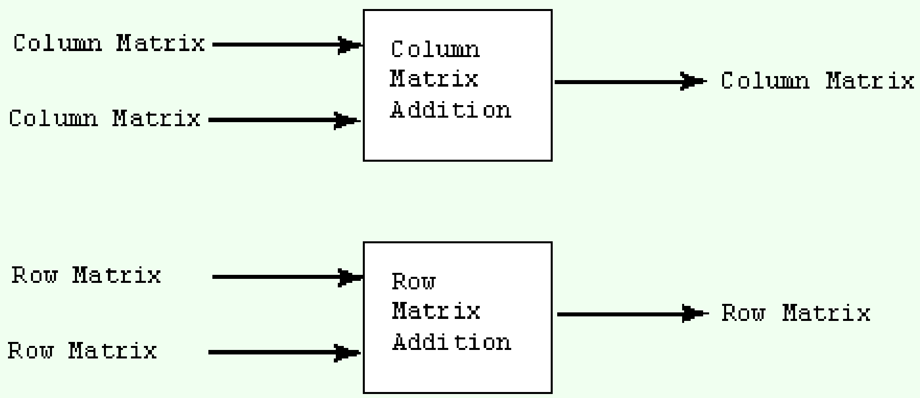
A good answer might be:

$$(1, 2, 3)^{T\,T} = (1, 2, 3)$$

Transposing twice gives you what you started with. This seems a bit dumb right now, but later on when you are doing algebraic manipulation it might help to remember this.

Column Matrix Addition

A column matrix added to another column matrix of the same dimension yields another column matrix (with the same dimension).



Addition is done by adding corresponding elements of the input matrices to produce each corresponding element of the output matrix. Row matrices are added in the same way.

$$(1, 2, 3) + (10, 20, 30) = (11, 22, 33)$$

$$(42, -12)^T + (8, 24)^T = (50, 12)^T$$

$$(9.2, -8.6, 3.21, 48.7) + (-2.1, 4.3, 1.0, 2.3) = (7.1, -4.3, 4.21, 51.0)$$

$$(32.98, -24.71, 9.392)^T + (-32.98, +24.71, -9.392)^T = (0, 0, 0)^T$$

If **a** and **b** are matrices of the same type, then **a** + **b** = **c** means that each element $c_i = a_i + b_i$

QUESTION 3:

Do the following problem:

$$(2, -2)^T + (8, 6)^T =$$

Click Here after you have answered the question

A good answer might be:

$$(2, -2)^T + (8, 6)^T = (10, 4)^T$$

Three Separate Problems

Your browser must be Java enabled to see this.

The matrices you add must be of the same dimension and same row or column type. When you add matrices, each dimension is handled independently of the others. For example, in adding two 3D column matrices, you have three separate additions using ordinary arithmetic. The first elements of the operand matrices are added together to make the first element of the result matrix.

Use the applet to play with matrix addition. Type a number into one of the boxes, hit Enter, and see the results. Make up a few easy problems for yourself, such as: "enter numbers into the first matrix so that each element of the result matrix is zero." Or, "if each element in the second matrix is zero, what will the result matrix look like?"

QUESTION 4:

Do the following two problems:

$$(8, 4, 6)^T + (2, -2, 9)^T =$$

$$(2, -2, 9)^T + (8, 4, 6)^T =$$

Click Here after you have answered the question

A good answer might be:

$$(8, 4, 6)^T + (2, -2, 9)^T = (10, 2, 15)^T$$

$$(2, -2, 9)^T + (8, 4, 6)^T = (10, 2, 15)^T$$

Commutative

You might suspect that the last problem is supposed to sneak in another math fact. You are right: Matrix addition is **commutative**. This means that

a + b = b + a.

This works for both row and column matrices of all dimensions. It is also true that:

a + b + c = b + c + a = c + a + b = ...

In other words, the order in which you add matrices does not matter. At least for addition, matrices work the same way as numbers, since, of course, 1 + 2 = 2 + 1. We will not always be so fortunate as to have matrices and numbers work the same way.

QUESTION 5:

Contemplate this problem:

$$(-1, -2, 3)^T + (1, 2, -3)^T =$$

Click Here after you have answered the question

A good answer might be:

```
( -1, -2, 3 )T  + ( 1, 2, -3 )T = ( 0, 0, 0 )T
```

Zero Column Matrix

Here is a problem which will probably not be on the midterm:

$$(73.6, -41.4)^T + (0.0, 0.0)^T = (73.6, -41.4)^T$$

A matrix with all zero elements is sometimes called a **zero matrix**. The sum of a zero matrix and a matrix **a** of the same type is just **a**.

In symbols the zero matrix is written as **0** (bold face zero) which is different than 0, the real number zero. Whether **0** is a row or column, and the number of elements in **0** depends on context.

QUESTION 6:

Do the following problem (which *might* be on the midterm):

$$(2, 5, -9) + (-32.034, 94.79, 201.062) + (-2, -5, 9) =$$

Click Here after you have answered the question

A good answer might be:

```
( 2, 5, -9 ) + ( -32.034, 94.79, 201.062 ) + ( -2, -5, 9 ) =

               ( -32.034, 94.79, 201.062 ) + ( -2, -5, 9 ) + ( 2, 5, -9 ) =
                                             ----------------------------

               ( -32.034, 94.79, 201.062 ) + ( 0, 0, 0)  =

               ( -32.034, 94.79, 201.062 )
```

Matrix Addition is Associative

When I give such a problem on a midterm, it is fun to watch students furiously add up the first two matrices, then furiously add in the third matrix.

Some students spoil my fun by realizing that (since matrix addition is commutative) the matrices can be rearranged into a more favorable order. Once the matrices are in a nice order, you can pick whichever "+" you want to do first. That last idea has a name:

> matrix addition is *associative*. This means that **(a + b) + c = a + (b + c)**.

This says "first add **a** to **b** then add that result to **c**." The result will be the same as if you did "add **a** to the result of adding **b** with **c**." This works for both row and column matrices of all dimensions.

QUESTION 7:

Thirty seconds to go on your midterm and you discover that you have left out a problem:

```
( 25.1, -19.6 ) + ( -5.0, 9.0 ) + ( 12.4, 8.92 ) + ( -20.1, 10.6 )  =
```

Can you get it done in time?

[Click Here after you have answered the question]

A good answer might be:

```
( 25.1, -19.6 ) + ( -5.0, 9.0 ) + ( 12.4, 8.92 ) + ( -20.1, 10.6 )  =
                  -------------                    ---------------
                     hmm...                          looks inviting...

                            \                         /
                              \                     /
                                \                 /
                                  \             /
                                    \         /
( 25.1, -19.6 ) + ( 12.4, 8.92 ) + ( -25.1, 19.6 )  = ( 12.4, 8.92 )
```

Some things which Cannot be Done

In computer science terms, the "+" symbol is *overloaded,* which means that the operation called for depends on the type of operands. For example:

1.34 + -9.06   <---- + means addition of real numbers

$(84.02, 90.31)^T + (-14.23, 10.85)^T$ <---- + means addition of 2D column matrices

The following have no meaning:

$34.5 + (84.02, 90.31)^T$ <---- can't add a number to a matrix

$(84.02, 90.31) + (-14.23, 10.85)^T$ <---- can't add two a row matrix to a column matrix

(84.02, 90.31) + (-14.23, 10.85, 32.75) <---- can't add matrices of different dimensions

These problems are clear when the elements are written out, as above, but it is less clear when variable symbols are used:

a + **x** <---- can't add a number to a matrix

x + **y** <----make sure that both are of the same type

There are some strange-looking things you can do, such as in the following:

QUESTION 8:

Perform the following addition:

$$(4.5, x_1, w) + (-2.3, 3, y_2) =$$

Click Here after you have answered the question

A good answer might be:

$$(4.5,\ x_1,\ w) + (-2.3,\ 3,\ y_2) = (2.2,\ x_1+3,\ w+y_2)$$

Matrix Subtraction

Two matrices of the same type can be subtracted to produce a third matrix of the same type. As you probably imagine, subtracting two matrices means subtracting the corresponding elements, *being careful to keep the elements in the same order:*

$$(10,\ 8,\ 12) - (2,\ 14,\ 9) = (10 - 2,\ 8 - 14,\ 12 - 9) = (8,\ -6,\ 3)$$

If **a** and **b** are matrices of the same type, **a** - **b** = **c** means that each element $c_i = a_i - b_i$

QUESTION 9:

Do these two problems:

$$(22,\ 5,\ -12) - (10,\ -5,\ 3) =$$

$$(10,\ -5,\ 3) - (22,\ 5,\ -12) =$$

Click Here after you have answered the question

A good answer might be:

```
(22, 5, -12 ) - (10, -5, 3 ) = ( 22 - 10, 5 - (-5), -12 - 3 ) = ( 12, 10, -15 )

(10, -5, 3 ) - (22, 5, -12 ) = ( 10 - 22, (-5) - 5, 3 - (-12) ) = (-12, -10, 15 )
```

Not Commutative

Another Math Fact emerges:

> Matrix subtraction is *NOT commutative*. This means that you can't change the order when doing **a - b**.

The **negative** of a matrix is this:

> **-a** means negate each element of **a**.

So, for example:

$$-(22,\ 5,\ -12\ ) = (\ -22,\ -5,\ 12\ )$$

$$-(\ 19.2,\ 28.6,\ 0.0\ )^T = (\ -19.2,\ -28.6,\ 0.0\ )^T$$

In symbols: if **a** is $(a_0, a_1, ..., a_2)$ then **-a** means $(-a_0, -a_1, ..., -a_2)$.

QUESTION 10:

Perform the following subtraction:

$$(\ -7.2,\ -98.6,\ 0.0\ )^T - (\ -2.2,\ -2.4,\ 3.0\ )^T$$

Click Here after you have answered the question

A good answer might be:

```
( -7.2, -98.6, 0.0 )T - ( -2.2, -2.4, 3.0 )T =
( -7.2 -(-2.2), -98.6 -(-2.4), 0.0 - 3.0)T =
( -5.0, -96.2,  - 3.0)T
```

Moving "-" Inside

If your brain works like mine (and you can't afford the cure), then you may find the following an easier way to do the problem:

```
( -7.2, -98.6, 0.0 )T - ( -2.2, -2.4, 3.0 )T =
( -7.2, -98.6, 0.0 )T + ( +2.2, +2.4, -3.0 )T =
( -5.0, -96.2,  - 3.0)T
```

The "outside -" was used to negate the second matrix, then the resulting two matrices were added. In symbols:

```
a - b = a + (-b)
```

This can be more useful to remember than it at first appears. You can negate matrices so the problem is one of matrix addition, then rearrange the addition (because addition is commutative):

```
a - b + c - d  = a + (-b) + c + (-d)
               = (-d) + a + c + (-b)
               = any rearrangement you want
```

Some other facts are:

```
a + (-a) = 0
a - a = 0
```

Notice that the **0** means the *zero matrix*, the matrix of the same type as **a**, but with all elements zero.

QUESTION 11:

Do the following operation:

```
( 4, -5, 6.2 ) + ( -43.132, 13.6, 86.5 ) - ( 4, -5, -4.8 ) =
```

Click Here after you have answered the question

```
( 4, -5, 6.2 ) + ( -43.132, 13.6, 86.5 ) - ( 4, -5, -4.8 ) =
```

A good answer might be:

This looks like another trap. Rather than blindly rushing in and calculate, try rearranging things:

```
( 4, -5, 6.2 ) + ( -43.132, 13.6, 86.5 ) - ( 4, -5, -4.8 ) =

( 4, -5, 6.2 ) + ( -43.132, 13.6, 86.5 ) + ( -4, 5, 4.8 ) =

( 4, -5, 6.2 )  + ( -4, 5, 4.8 ) + ( -43.132, 13.6, 86.5 ) =

( 0, 0, 11 ) + ( -43.132, 13.6, 86.5 ) = (-43.132, 13.6, 97.5 )
```

You would probably skip a few steps if you were doing this mentally.

More Practice

Do a few more practice problems before you move on.

QUESTION 12:

What is the sum of the following three displacements:

$$\mathbf{d} = (-12.4, 14.8, 0.0)^T$$
$$\mathbf{e} = (6.2, -10.2, 17.0)^T$$
$$\mathbf{f} = (6.2, -4.6, -17.0)^T$$

Click Here after you have answered the question

A good answer might be:

```
d = ( -12.4,  14.8,   0.0 )T
e = (   6.2, -10.2,  17.0 )T
f = (   6.2,  -4.6, -17.0 )T
    ---------------------
    (   0.0,   0.0    0.0 )T
```

Yet More Practice

QUESTION 13:

Find x, y, and z so that the following is true:

```
a = ( 8.6, 7.4, 3.9 )
b = ( 4.2, 2.2, -3.0 )
c = ( x, y, z )

a + b + c = 0
```

Click Here after you have answered the question

Find x, y, and z so that the following is true:

$$\mathbf{a} = (8.6, 7.4, 3.9)$$
$$\mathbf{b} = (4.2, 2.2, -3.0)$$
$$\mathbf{c} = (x, y, z)$$

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$

A good answer might be:

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = (12.8, 9.6, 0.9) + (x, y, z) = (12.8+x, 9.6+y, 0.9+z) = (0, 0, 0)$$

So it must be that:

$$12.8+x = 0$$
$$9.6+y = 0$$
$$0.9+z = 0$$

So: x = -12.8, y = -9.6, z = -0.9

Oh No! Algebra

Yes, algebra. The problem was: find the elements of **c** when

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$

If you didn't know they were matrices, you might have been tempted to work this using real number algebra:

$$\begin{aligned} \mathbf{a} + \mathbf{b} + \mathbf{c} &= \mathbf{0} \\ \mathbf{a} + \mathbf{b} &= -\mathbf{c} + \mathbf{0} \\ (\mathbf{a} + \mathbf{b}) &= -\mathbf{c} \\ -(\mathbf{a} + \mathbf{b}) &= \mathbf{c} \end{aligned}$$

In fact, this works. As long as every matrix is of the same type, and the operations are only "+" or "-", you can pretend that you are doing ordinary algebra. Notice that the last equations means "add **a** with **b**, negate the result, then the result is equal to **c**."

To see this, look at just the first elements of the matrices:

$$\begin{aligned} (a_0, \ldots) + (b_0, \ldots) + (c_0, \ldots) &= (0, \ldots) \\ (a_0, \ldots) + (b_0, \ldots) &= -(c_0, \ldots) + (0, \ldots) \\ (a_0, \ldots) + (b_0, \ldots) &= -(c_0, \ldots) \\ (a_0 + b_0, \ldots) &= -(c_0, \ldots) \\ -(a_0 + b_0, \ldots) &= (c_0, \ldots) \\ -(a_0 + b_0) &= c_0 \end{aligned}$$

If this is too ugly for you this early in the morning, mentally erase some of the junk:

$$\begin{aligned} a_0 + b_0 + c_0 &= 0 \\ a_0 + b_0 &= -c_0 + 0 \\ a_0 + b_0 &= -c_0 \\ a_0 + b_0 &= -c_0 \end{aligned}$$

$-(a_0 + b_0) = c_0$

Of course the other elements follow the same pattern so the result is true for the matrix as a whole.

QUESTION 14:

Find c_0 and c_1 so that the following is true:

$\mathbf{a} = (-4, 2)^T$
$\mathbf{b} = (8, 3)^T$
$\mathbf{c} = (c_0, c_1)^T$

$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

Click Here after you have answered the question

Find c_0 and c_1 so that the following is true:

$$\mathbf{a} = (-4, 2)^T$$
$$\mathbf{b} = (8, 3)^T$$
$$\mathbf{c} = (c_0, c_1)^T$$

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$

A good answer might be:

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$
$$\mathbf{a} + \mathbf{b} = \mathbf{-c}$$
$$(4, 5)^T = -(c_0, c_1)^T$$
$$-(4, 5)^T = (c_0, c_1)^T$$
$$(-4, -5)^T = (c_0, c_1)^T$$
$$c_0 = -4$$
$$c_1 = -5$$

End of the Chapter

If you have gotten this far, you may have actually learned something:

- Transpose of a column or row matrix.
- Addition of column and row matrices.
- Commutative property of matrix addition.
- Associative property of matrix addition.
- Subtraction of column and row matrices.
- Non-commutative property of matrix subtraction.

Click on a term to see where it was discussed in this chapter. Remember to click on the "Back" arrow of your browser to get back to this page. The next chapter will suggest ways in which what you have learned may be useful.

Click here to go back to the main menu.

You have reached the End.

created 09/29/97; revised 08/06/00

CHAPTER 3 -- Vector Addition

The previous chapter discussed how to do column and row matrix addition. This is an easy, mechanical procedure. This chapter will discuss why it is a *useful* procedure as well by discussing how it is used to represent the *geometrical* operation of vector addition.

- Addition of geometrical vectors.
- Addition of vectors using the head-to-tail rule.
- Coordinate frames.
- Representing geometrical points with column matrices.
- Representing geometrical vectors with column matrices.
- Addition of vectors using column matrices.
- Vector plus point addition.

QUESTION 1:

Just for practice, add the following two column matrices:

$\mathbf{a} = (3, 2)^T$
$\mathbf{b} = (-2, 1)^T$

Click Here after you have answered the question

```
a = ( 3, 2 )T
b = ( -2, 1 )T
```

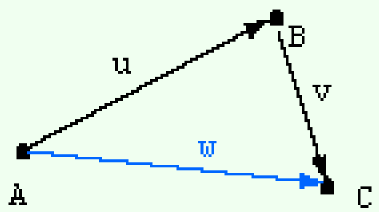
A good answer might be:

```
a + b = c = ( 1, 3 )T
```

Vectors in 2D Space

Vectors are geometric objects. A particular vector can be represented in a variety of ways. It is important to understand the difference between a geometric object (such as a point or a vector) and the way it is represented (often with a column matrix).

Recall that in computer graphics it is common for a virtual world to be modeled with points a vectors. As the virtual tourist (or the "hero" of a first-person computer game) wanders through the world it does not change (much), but the viewpoint and resulting 2D image changes greatly. The geometric points and vectors do not change, but as the viewpoint changes, their representations as column matrices changes constantly.



The diagram shows points A, B, and C (in two dimensions). Vector u is the displacement from A to B, and vector v is the displacement from B to C. A displacement is a distance and a direction. The distance and direction from A to B is the vector u.

The displacement from B to C is the vector v. And the displacement from A to C is the vector w. The effect of moving through the displacement u and then through the displacement v is the same as moving through the displacement w. In symbols:

u + v = w

QUESTION 2:

Does the addition depend on location?

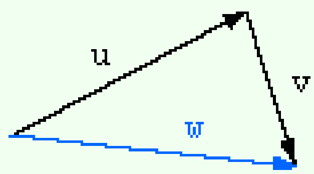
Click Here after you have answered the question

Does the addition depend on location?

A good answer might be:

No, because vectors have no position.

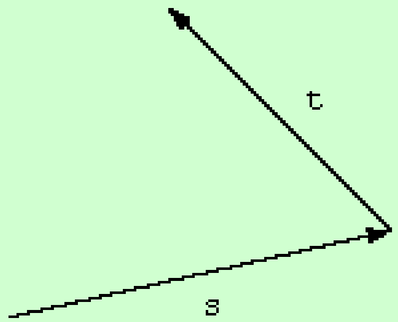
Position Independent



The diagram included points to make it clear how vector addition is used. But more commonly vector addition is shown as in the diagram at left. The diagram shows that the effect of moving through the displacement u and then moving through the displacement v is the same as moving through the displacement w, no matter where you start from.

For example, say that you wish to move all the points of a geometric object through displacement u and then through displacement v. You could do this by moving each point through displacement w (ie. w does not depend on position).

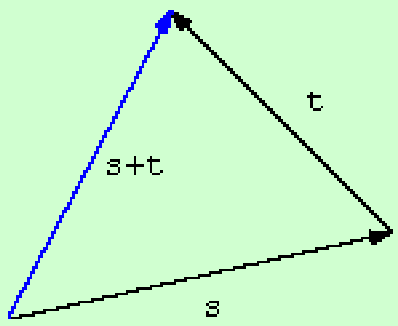
QUESTION 3:



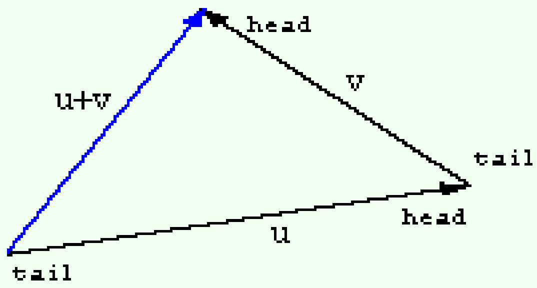
Mentally draw the line in the diagram that represents the sum of vector s with vector t.

Click Here after you have answered the question

A good answer might be:



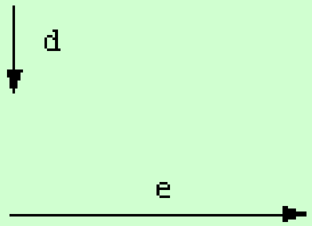
Head-to-Tail Rule



The rule for adding vector u to vector v is:

Head-to-Tail Rule: Move vector v (keeping its length and orientation the same) until its tail touches the head of u. The sum is the vector from the tail of u to the head of v.

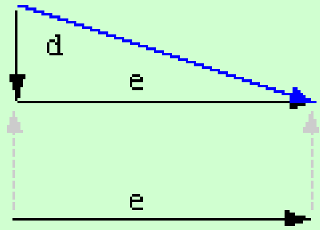
QUESTION 4:



Mentally add vector e to vector d.

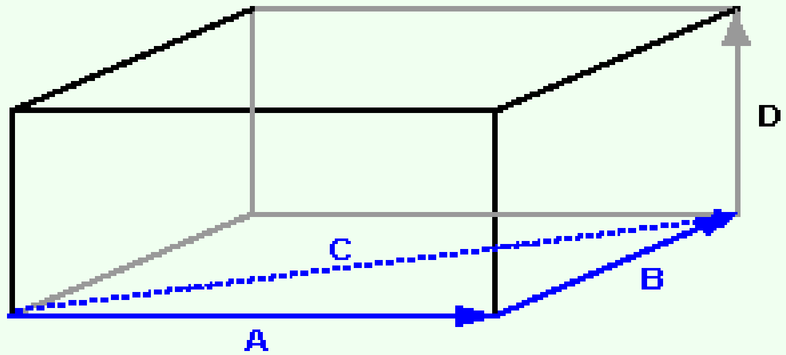
Click Here after you have answered the question

A good answer might be:



The diagram shows sliding the vector e until its tail touches the head of d. The result is the vector from the tail of d to the head of e.

Vector Addition in 3D



The diagram shows 3D vectors a and b added to form c (the 3D figure is there to aid in visualizing the three dimensions). The head-to-tail rule works in 3D as well.

When two vectors are added in 3D, the head-to-tail rule defines a triangle oriented in 3D space.

QUESTION 5:

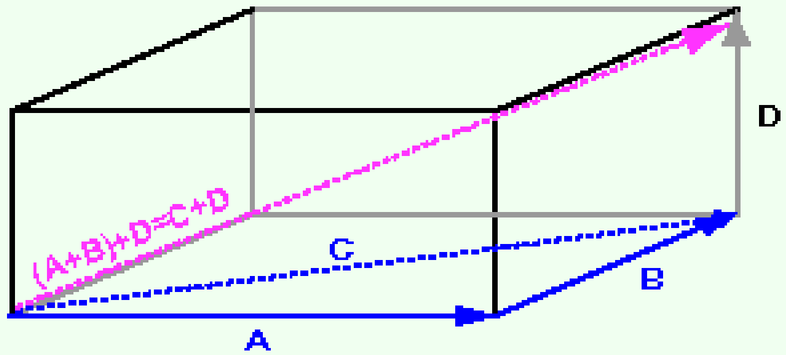
Is it possible (do you think) to add three vectors together, like a+b+d in the figure?

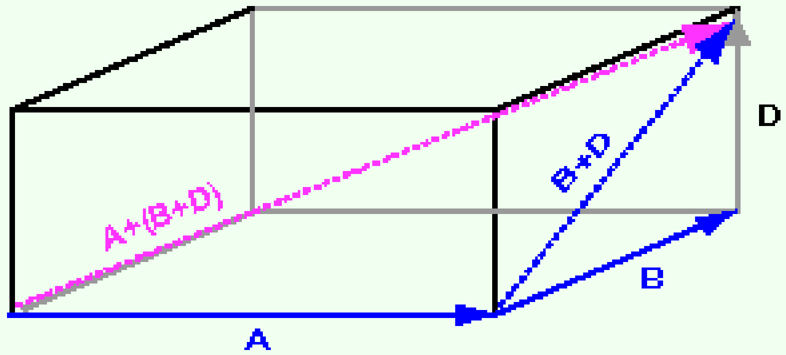
Click Here after you have answered the question

A good answer might be:

Yes. The result is the total displacement that would result in following each vector in turn.

Associative





The top diagram shows the result of adding (a+b) + d = c + d. The result is the vector with length and direction the same as the diagonal of the figure.

The bottom diagram shows the result of adding a + (b+d). The result is the same. This is a demonstration of the **associative** property of vector addition:

a + (b + c) = (a + b) + c

The rule works in all dimensions. Mostly the geometric vectors of computer graphics are two dimensional and three dimensional (although, in general, higher dimensions are possible).

The associative property means that sums of several vectors can be written like a + b + c + d + e without parentheses.

QUESTION 6:

Say that you walk five blocks north, and then three blocks east. Will you end up at the same place if you walk three blocks east and then five blocks north?

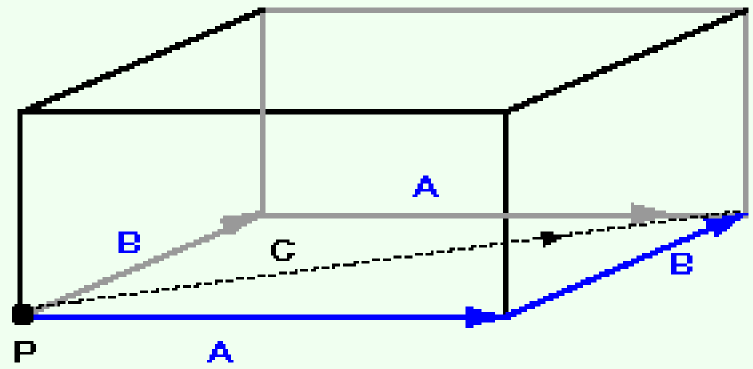
Click Here after you have answered the question

Say that you walk five blocks north, and then three blocks east. Will you end up at the same place if you walk three blocks east and then five blocks north?

A good answer might be:

Yes.

Commutative



Vector addition is commutative, just like addition of real numbers. That is:

Commutative Property: a+b = b+a

If you start from point P you end up at the same spot no matter which displacement (a or b) you take first.

Notice how in the diagram the head-to-tail rule yields vector c for both a+b and b+a.

QUESTION 7:

Does the commutative property hold in all dimensions?

Click Here after you have answered the question

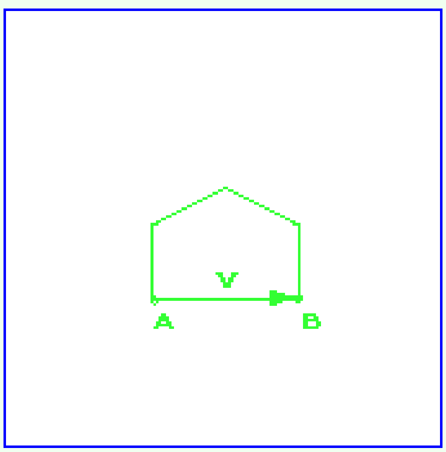
Does the commutative property hold in all dimensions?

A good answer might be:

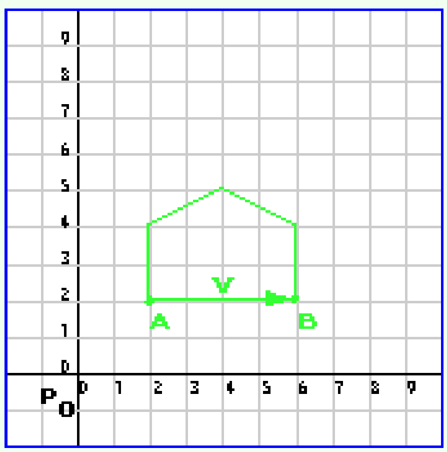
Yes.

Representing Points with Column Matrices

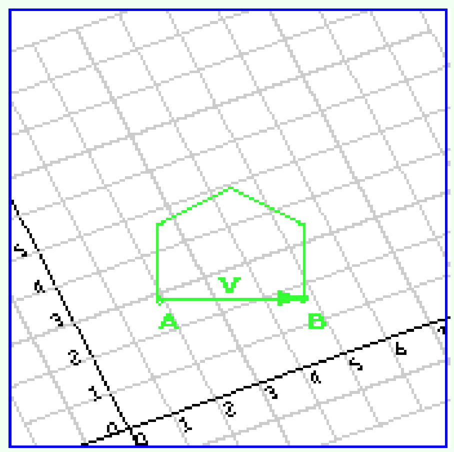
Points can be represented with column matrices. To do this, you first need to decide on a **coordinate frame** (sometimes called just *frame*. A coordinate frame in consists of a distinguished point P_0 (called the origin) and an axis for each dimension (often called "x" and "y"). In 2D space there are two axes; in 3D space there are three axes (often called "x", "y", and "z").



Virtual World



Virtual World with Coordinate Frame



Virtual World with a different Coordinate Frame

For now, let us talk about 2D space. The left diagram shows a (rather simple) virtual world in 2D space. The points and vectors exist in the space independent of any coordinate frame. The next diagram shows the **same** virtual world, this time with a coordinate frame consisting of a particular point P_0 and two axes. In this coordinate frame, the point A is represented by the column matrix $(2, 2)^T$.

The right diagram shows the **same** virtual world, but with a different coordinate frame. In the second coordinate frame, the point A is represented by the column matrix $(2, 3)^T$.

QUESTION 8:

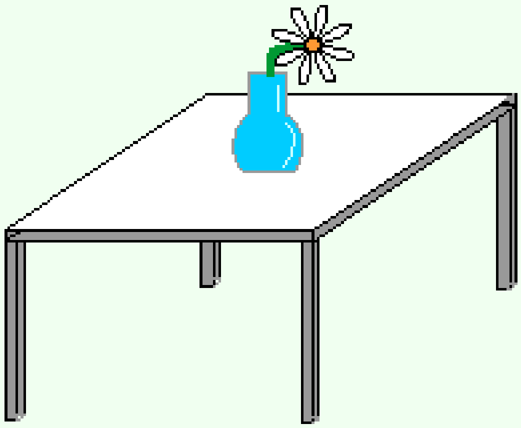
- How is point B represented using the first coordinate frame?
- How is point B represented using the second coordinate frame?

Click Here after you have answered the question

A good answer might be:

- How is point B represented using the first coordinate frame?
  - $(6, 2)^T$
- How is point B represented using the second coordinate frame?
  - $(4.8, 1.6)^T$

Representing Points with Different Frames



The vital (and often confusing) idea is that there is one virtual world, but any number of coordinate frames being used at any moment. Often there are many frames being used for different objects and different viewpoints. This is something you already do all the time with the real world. The hard part is seeing how it works with a virtual world.

For example, say you have a vase set on a table. You could describe its location as "8 inches from the front edge of the table and 12 inches from the right edge."

Or, you could pick some other frame and say "17 inches from the left edge of the table and 38 inches from the back edge." Or you could describe where it is in relation to the foot of the left front leg of the table, or....

QUESTION 9:

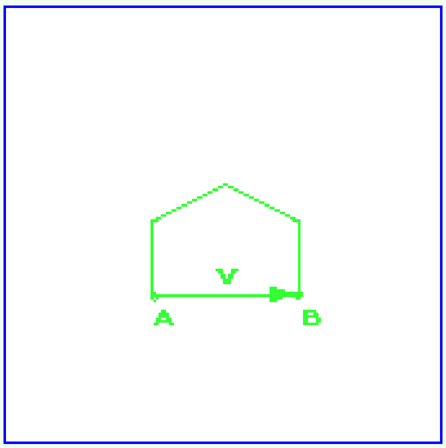
As you are out for a walk, a stranger approaches you and asks for directions to the post office. Which coordinate frame will you use: latitute and longitude, or number of city blocks left and right of your current position?

Click Here after you have answered the question

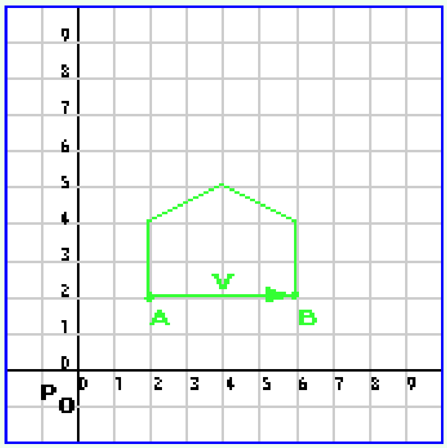
A good answer might be:

Unless the stranger has a GPS receiver, the city block coordinate frame is likely to be the most useful.

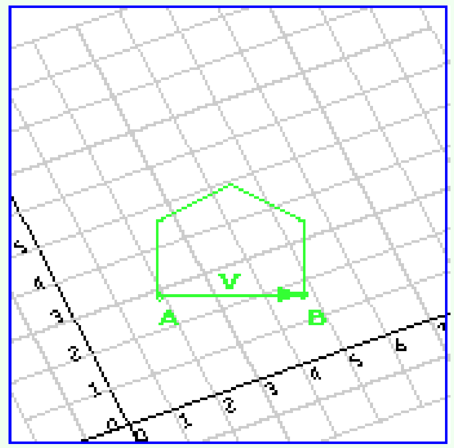
Representing Vectors with Column Matrices



Virtual World



Virtual World with Coordinate Frame



Virtual World with a different Coordinate Frame

Back to our exciting virtual world. The displacement from point A to point B is the vector v. The vector does not depend on any coordinate system. Using the first coordinate system (middle diagram) the displacement is represented as

```
(difference in x coordinates, difference in y coordinates)
```

In the diagram this is $(4, 0)^T$. Moving 4 units from point A in the direction of the x axis brings us to point B. In the second coordinate system (right diagram) the same vector is represented by $(2.8, -1.4)^T$.

QUESTION 10:

Points and vectors both are represented with column matrices. Is this likely to be confusing?

Click Here after you have answered the question

A good answer might be:

Yes. In your graphics text this confusion leads to the use of *homogeneous coordinates*, a different way of representing points and vectors with column matrices.

Vector Addition represented by Column Matrix Addition

But for these notes we will continue to use column matrices as explained above. Later on these ideas will be extended to a different method of representing points and vectors.

If vectors are represented with column matrices, then vector addition is represented by addition of column matrices. For example:

$$\mathbf{a} = (3, 2)^T$$
$$\mathbf{b} = (-2, 1)^T$$
$$\mathbf{a} + \mathbf{b} = \mathbf{c} = (1, 3)^T$$

The diagram shows the head-to-tail rule used to add a and b to get c. Adding the column matrices **a** and **b** yields the column matrix **c**. This matrix is the correct representation of the vector c.

QUESTION 11:

Now compute **c** = **b + a** for the same **a** and **b** as above.

Click Here after you have answered the question

A good answer might be:

$\mathbf{b} + \mathbf{a} = \mathbf{c} = (1, 3)^T$

Different Order, but Same Answer

To draw the diagram for this follow the rules, with **b** as the starting vector:

1. Draw the first vector **b** as an arrow with its tail at the origin.
2. Draw the second vector **a** as an arrow with its tail at the point of the first.
3. The sum is the arrow from the origin to the tip of the second vector.

(Remember that *vectors have no location* so you can draw a picture of a vector with its tail starting on any point you want.)

Since vector addition is commutative, the result must be the same.

QUESTION 12:

(Review:) Are vector addition and column matrix addition both *associative*?

Click Here after you have answered the question

(Review:) Are vector addition and column matrix addition both *associative*?

A good answer might be:

Yes.

Practice

Now it is your turn. Here are two vectors:

$$\mathbf{r} = (4, 3)^T$$
$$\mathbf{s} = (1, 2)^T$$

form the sum:

$$\mathbf{t} = \mathbf{r} + \mathbf{s}$$

Do it by column matrix addition, and by the head-to-tail rule. In the applet, put the mouse pointer on the point where you wish to start an arrow, left-click and drag to the point where you wish the arrow to end.

QUESTION 13:

When your picture is complete, go on to the next page.

Click Here after you have answered the question

A good answer might be:

Applying the rule:

$$\mathbf{t} = (5,\ 5)^T$$

Same Problem

Now add up the vectors in the opposite order: form the sum

$$\mathbf{t} = \mathbf{s} + \mathbf{r}$$

where

$$\mathbf{r} = (\ 4,\ 3\ )^T$$
$$\mathbf{s} = (\ 1,\ 2\ )^T$$

Do this by adding two new arrows to the above diagram. (Remember, vectors have no location, so it is OK to have several arrows in a diagram for the same vector.)

QUESTION 14:

Continue to the next page when your answer is finished.

Click Here after you have answered the question

A good answer might be:

As before,

```
T = (5, 5)^T
```

Parallelogram for Vector Addition

There are two ways to form the sum:

```
T = S + R
T = R + S
  where
R = ( 4, 3 )^T
S = ( 1, 2 )^T
```

There are two ways to draw the diagram, depending on which arrow's tail you put at the origin. If you draw both versions, then you get a parallelogram with the sum of the vectors as the diagonal arrow who's tail starts at the origin.

What is a parallelogram? you might ask, if your high school geometry is a bit murky. A parallelogram is a four sided figure with opposite sides parallel and equal in length. So, for example, the blue arrows representing the vector **s** are the same length and same direction. The green arrows representing the vector **r** have their same length and same direction.

QUESTION 15:

u = $(-3, 2)^T$, **v** = $(1, -5)^T$ form the sum: **w** = **u** + **v**

Click Here after you have answered the question

u = $(-3, 2)^T$, **v** = $(1, -5)^T$ form the sum: **w** = **u** + **v**

A good answer might be:

w = $(-2, -3)^T$

More Practice

Now draw **u**, **v**, and **w** on the graph paper, where, as before, **u** = $(-3, 2)^T$, **v** = $(1, -5)^T$, and **w** = $(-2, -3)^T$

QUESTION 16:

Continue to the next page when you have answered.

Click Here after you have answered the question

A good answer might be:

Your answer should look like the diagram.

Two Legged Trip

The diagram shows **w** = **u** + **v**, where **u** = (-3, 2)^T, **v** = (1, -5)^T, and **w** = (-2, -3)^T

Now consider an ant that starts at the origin and walks along vector **u**, then walks along vector **v**, to end up at the tip of **w**.

A second ant starts at the origin and walks along vector **w** to its end. Both ants end up at the same point.

QUESTION 17:

Did the two ants walk the same distance?

Click Here after you have answered the question

Did the two ants walk the same distance?

A good answer might be:

No. In this case, it is clear that walking in a straight line to the final destination is shorter.

Summing Displacements =/= Summing Lengths

Summing displacements (vectors) is not the same as summing their length. In the previous example, the sum of **u** and **v** yields a vector that is shorter than the length of **u** plus the length of **v**. For now, just observe this by looking at the previous diagram, and remembering that "a straight line is the shortest distance between two points." (Later on this will be discussed using the Pythagorean Formula.)

This fact is called the **triangle inequality**:

```
length( u + v ) <= length( u ) + length( v )
```

Here is a case where the length of the sum is *much* shorter than the sum of the lengths:

$$\mathbf{e} = (5, 4)^T$$
$$\mathbf{g} = (-4.9, -3.9)^T$$
$$\mathbf{e} + \mathbf{g} = (.1, .1)^T$$

(For clarity **g** has been moved slightly moved away from where it should be.)

QUESTION 18:

Can you think of a situation where the length of the result is equal to the sum of the length of the two input vectors?

Click Here after you have answered the question

Can you think of a situation where the length of the result is equal to the sum of the length of the two input vectors?

A good answer might be:

This will be true if one vector is in the same direction as the other.

Vectors in the Same Direction

This situation is illustrated in the diagram. The head-to-tail rule is used to sum two vectors pointing in the same direction. (Vectors in the same direction are called **colinear**). The sum vector is slighly moved out of its correct position so that you can see it. The summed vectros are $(4, 3)^T$ and $(2, 1.5)^T$. The sum is $(6, 4.5)^T$

For now, to determine if two vectors have the same orientation you have to look at a picture or use geometry. Later on there will be a procedure to test if two vectors point in the same direction.

This example is a case where the "=" sign in the formula applies:

length(**u** + **v**) <= length(**u**) + length(**v**)

QUESTION 19:

Can you think of *another* case in which the "=" sign applies?

Click Here after you have answered the question

Can you think of *another* case in which the "=" sign applies?

A good answer might be:

When one of the vectors is a zero vector:

length(**u** + **0**) <= length(**u**) + length(**0**)

Diagram for Adding the Zero Vector

Add the zero matrix $(0, 0)^T$ to the matrix $(4, 3)^T$. If $(0, 0)^T$ represents a displacement vector, it means do nothing to change position. In the diagram, no arrow can be drawn for the zero vector since it has no length.

QUESTION 20:

Now consider what this might mean: add the vector $(1, 2)^T$ to the point $(4, 4)^T$.

If you think that makes sense, then do the calculation. What type of object is the result?

Click Here after you have answered the question

Add the vector $(1, 2)^T$ to the point $(4, 4)^T$.

A good answer might be:

The result is a point: $(4, 4)^T + (1, 2)^T = (5, 6)^T$

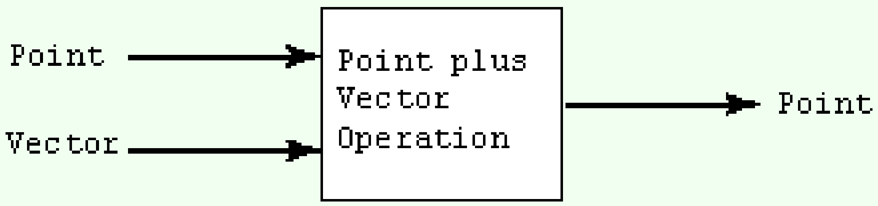
Adding Points and Vectors

If the vector $(1, 2)^T$ is a displacement (ie., an amount by which to change x, and an amount by which to change y), then the result must be a point in a new location:

$$(4, 4)^T + (1, 2)^T = (5, 6)^T$$

This is one of the situation where using the same representation (ie. column matrices) for both points and vectors is confusing. You have to keep track of what type of object each matrix represents.

A displacement vector added to a point results in a point. Here is a diagram of that:



This is slighly odd. Two mathematical objects *of different types* are added together. Again, the "+" sign is overloaded.

Here is a diagram showing the point (4, 4). Draw the vector $\mathbf{(1, 2)^T}$ as a displacement from that point to the sum (5, 6).

QUESTION 21:

Go on to the next page when you are done.

Click Here after you have answered the question

A good answer might be:

Your picture should look like the one below.

Translation

Sometimes the operation of adding a vector to a point is called **translation**. The original point is sometimes said to have been "translated to a new location."

The vocabulary here might be confusing. In mathematical terms, the sum of a vector and a point yields a new point. The first point remains unchanged. But in computer graphics terms it is nice to think about pictures "moving across the screen" when displacement vectors are added to their points.

QUESTION 22:

We have seen:

- vector + vector
- point + vector
- point - point

Do you suppose that this operation can be done:

- point + point

Click Here after you have answered the question

A good answer might be:

The sum of two points does not mean anything, geometrically. The sum of two column vectors that represent points is possible, mechanically, but is meaningless.

End of Chapter

You have reached the end of this chapter. Perhaps you would like to see if you remember any of it:

- Head to tail rule for geometric vector addition.
- Head to tail rule in three dimensions.
- Associative property of vector addition.
- Commutative property of vector addition.
- Coordinate frames.
- Representing points using coordinate frames and column matrices.
- Representing points using coordinate frames and column matrices.
- Parallelogram in vector addition
- Triangle inequality
- Colinear vectors.
- Zero vector addition.
- Point plus vector addition.

The next chapter will further discuss operations on vectors. The chapter after that will discuss further operations on vectors. The chapter after *that* will further discuss further operations on vectors. It is going to be a long semester.

Click here to go back to the main menu.

You have reached the End.

created 08/26/97; revised 08/10/00

CHAPTER 4 -- Vector Length

This chapter will discuss the *length* of vectors and how length is computed using the column matrix representation of vectors. The next chapter will discuss another vector property, *direction*. Vectors of all dimensions have length and direction. But to make the discussion easier to visualize most of the examples in this chapter use vectors in 2D space.

- Length of 2D and 3D vectors.
- Pythagorean formula.
- Calculating the length of a vector from its column matrix representation.
- Length of the zero vector.
- Length of a negated vector.

QUESTION 1:

What do you suppose is the length of the vector represented by: $(3, 0)^T$?

Click Here after you have answered the question

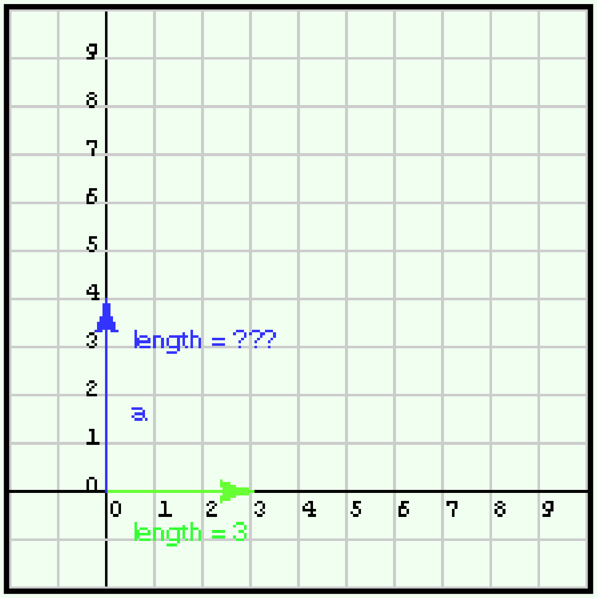
What do you suppose is the length of the vector represented by:

$(3, 0)^T$

A good answer might be:

It seems plausible that the length is 3 units.

Vectors Aligned with X-Y Axes



When a 2D vector is aligned with the x axis the column matrix that represents it has a non-zero value in the first element, and zero for the second element. The green vector is represented by: $\mathbf{(3, 0)^T}$. It is easy to determine its length---the length is the value of the single non-zero element.

Vector **a** (the blue vector in the diagram) is aligned with the y axis. Its matrix is $\mathbf{(0,4)^T}$.

QUESTION 2:

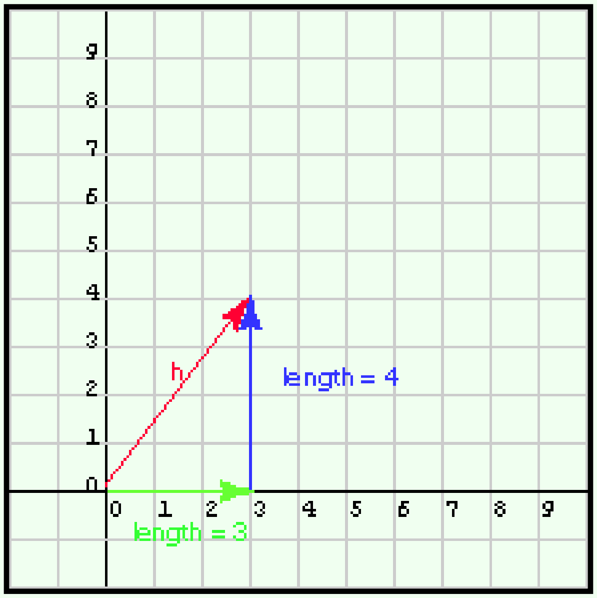
What is the length of **a**?

Click Here after you have answered the question

A good answer might be:

a = (0,4)^T is aligned with the Y axis. Its length, 4 units, can be read off the diagram.

Pythagorean Formula



Of course, vectors have no fixed location, so vector **a** can be drawn anywhere. The diagram shows the vectors of length 3 and 4, and with a new vector, **h**.

The length of vector **h** is harder to figure out. But not much harder, especially if you know about "3, 4, 5 right triangles." The three vectors can be arranged into a right triangle with **h** as the hypotenuse. The other sides are 3 and 4, so the length of **h** is 5.

You probably know the Pythagorean Formula:

(length of hypoteneuse)^2 = (length of first side)^2 + (length of second side)^2

Using this with a right triangle with sides of 3 and 4:

(length of hypoteneuse)^2 = $3^2 + 4^2$

(length of hypoteneuse)^2 = $9 + 16 = 25$

(length of hypoteneuse) = 5

QUESTION 3:

What is the length of the hypoteneuse of a right triangle whose two sides are 6 and 8?

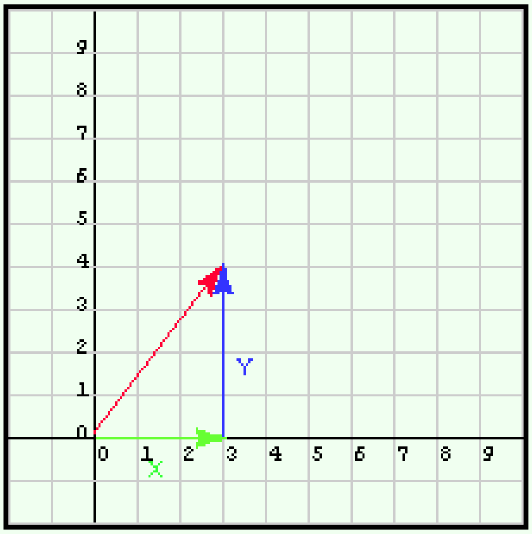
Click Here after you have answered the question

What is the length of the hypoteneuse of a right triangle whose two sides are 6 and 8?

A good answer might be:

$$\text{length}^2 = 6^2 + 8^2$$
$$\text{length}^2 = 36 + 64 = 100$$
$$\text{length} = 10$$

Formula for Vector Length



The formula for the length of a 2D vector is the Pythagorean Formula. Say that the vector is represented by **$(x, y)^T$**. Put the vector with its tail at the origin. Now make a triangle by drawing the two sides:

$$\text{side\_1} = \mathbf{(x, 0)^T}$$

$$\text{side\_2} = \mathbf{(0, y)^T}.$$

The length of side_1 is x, and the length of side_2 is y, so:

$$\text{length } \mathbf{(x, y)^T} = \sqrt{(x^2 + y^2)}$$

In this formula, $\sqrt{\ }$ means the positive square root. We don't (of course) want the length to be negative.

QUESTION 4:

What is the length of the vector represented by **$(4, 3)^T$** ?

Click Here after you have answered the question

A good answer might be:

The length is 5.0. You could use the formula, or by realize that this is another 3-4-5 right triangle.

Symbol for Vector Length

The formula works for vectors aligned with the axes:

```
length of( (8,0)^T )   =   √( 8*8 + 0*0 ) = 8
```

It is awkward to keep saying "length of ( )." There is a symbol for this:

length of (**a**) = **| a |**

Your browser might not show this well. The symbol *a* has a vertical bar on each side. Sometimes books will use two vertical bars on each side. Using the new notation:

$$|(x, y)^T| = \sqrt{x^2 + y^2}$$

QUESTION 5:

If **g** = $(1, 1)^T$, what is **| g |**?

Click Here after you have answered the question

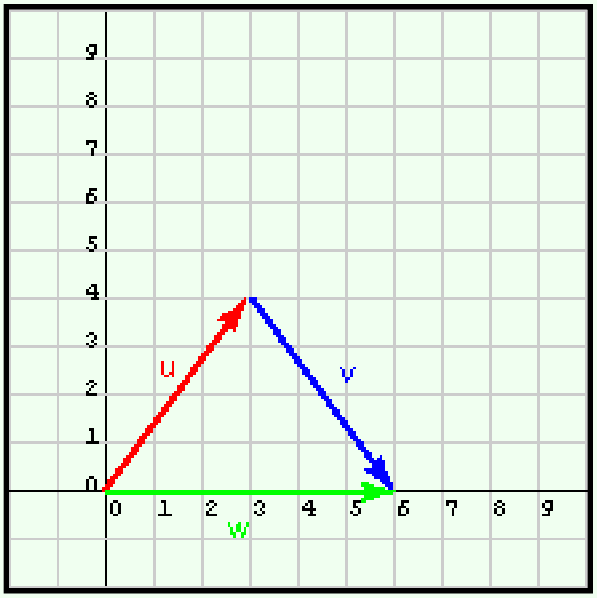
If **g** = $(1, 1)^T$, what is | **g** |?

A good answer might be:

Plugging into the formula: | **g** | = $\sqrt{(1^2 + 1^2)}$ = $\sqrt{2.0}$ = 1.414213562373

Notice that the length is NOT 1.0, nor 2.0. People sometimes make that mistake if they are not careful.

Demonstration the Triangle Inequality



Remember from the previous chapter that the length of the sum of two vectors is less than or equal to the sum of the lengths of the two vectors:

length of (**u** + **v**) <= length of (**u**) + length of (**v**)

In our new notation:

| **u** + **v** | <= | **u** | + | **v** |

Consider the vectors represented by:

$$\mathbf{u} = (3, 4)^T$$
$$\mathbf{v} = (3, -4)^T$$
$$\mathbf{w} = \mathbf{u} + \mathbf{v} = (6, 0)^T$$

The picture shows how the triangle inequality works for this example. But does it work numerically?

QUESTION 6:

What is **| u |**, **| v |**, and **| w |** ?

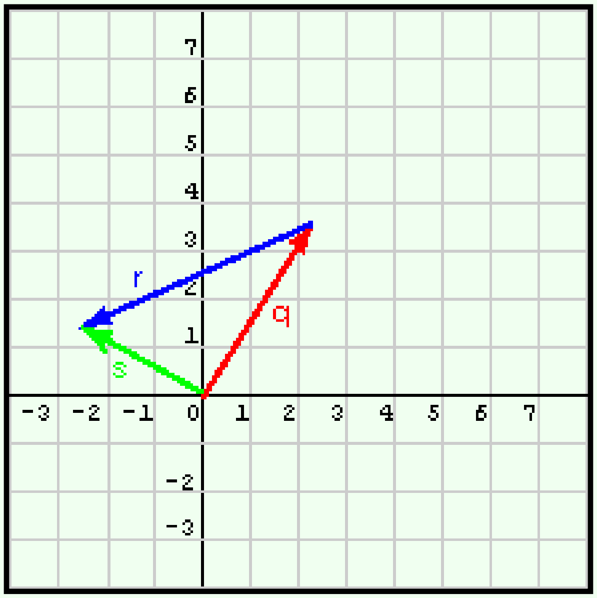
Click Here after you have answered the question

A good answer might be:

$$\mathbf{u} = (3, 4)^T \quad |\mathbf{u}| = 5$$
$$\mathbf{v} = (3, -4)^T \quad |\mathbf{v}| = 5$$
$$\mathbf{w} = \mathbf{u} + \mathbf{v} = (6, 0)^T \quad |\mathbf{w}| = 6$$

And 6 < 5 + 5, demonstrating that **| u + v |** <= **| u |** + **| v |**

Another Example



That was too easy. Usually vector elements are not convenient integer values. Here are some more realistic vectors, represented by the column matrices:

$$\mathbf{q} = (2.2, 3.6)^T$$
$$\mathbf{r} = (-4.8, -2.2)^T$$
$$\mathbf{s} = \mathbf{q} + \mathbf{r}$$

Of course, the pythagorean formula still applies:

$$|\,(x, y)^T\,| = \sqrt{(x^2 + y^2)}$$

But now you will have to get out a calculator (or click on the calculator in Windows).

QUESTION 7:

What is **| q |**, **| r |**, and **| s |** ?

[Click Here after you have answered the question]

q = **$(2.2, 3.6)^T$**
r = **$(-4.8, -2.2)^T$**
s = **q + r**

A good answer might be:

$\|\mathbf{q}\|$	=	$\sqrt{(2.2*2.2 + 3.6*3.6)}$	=	$\sqrt{(4.84 + 12.96)}$	=	$\sqrt{17.8}$	=	4.219
$\|\mathbf{r}\|$	=	$\sqrt{(-4.8* -4.8 + -2.2 * -2.2)}$	=	$\sqrt{(23.04 + 4.84)}$	=	$\sqrt{27.88}$	=	5.280
$\|\mathbf{s}\|$	=	$\sqrt{(-2.6 * -2.6 + 1.4*1.4)}$	=	$\sqrt{(6.76 + 1.96)}$	=	$\sqrt{8.72}$	=	2.953

As expected, **| s |** is less than **| q |** + **| r |**.

Length of 3D Vectors

Three dimensional vectors have length. The formula is about the same. The length of a vector represented by a three-component matrix is:

$$\boxed{|\,(\mathbf{x}, \mathbf{y}, \mathbf{z})^{\mathbf{T}}\,| = \sqrt{(x^2 + y^2 + z^2)}}$$

For example:

$$|\,(\mathbf{1}, \mathbf{2}, \mathbf{3})^{\mathbf{T}}\,| = \sqrt{(1^2 + 2^2 + 3^2)} = \sqrt{(1 + 4 + 9)} = \sqrt{14} = 3.742$$

QUESTION 8:

What is the length of **$(-1, -2, 3)^T$**

Click Here after you have answered the question

What is the length of **$(-1, -2, 3)^T$**

A good answer might be:

$$| (-1, -2, 3)^T | = \sqrt{(-1 * -1 + -2 * -2 + 3 * 3)} = \sqrt{(1 + 4 + 9)} = \sqrt{14} = 3.742$$

Squaring the elements of the vector results in a sum of all positive values, ensuring a positive (or zero) value for length.

Geometrical Vectors

Keep in mind that vectors are geometrical objects: a length and a direction in space. Vectors are represented with column matrices. The formulas for length that have been presented in this chapter assume that a coordinate frame is being used and that the vectors are represented with column matrices in that frame.

Your graphics text book will discuss how *homogeneous coordinates* are used to represent vectors. That method uses 4-component column matrices to represent vectors in three dimensions. Calculating the length of a vector represented in that manner will call for a slight modification of the formulas discussed here.

Don't worry terribly about that now. Details will come soon enough. But do take the time to become comfortable about the idea that column matrices such as we have been using are not the only way to represent vectors, and that length is a property of the vector, not of the column matrix that represents it.

QUESTION 9:

Is the distinction between an object and its representation of any importance in computer science?

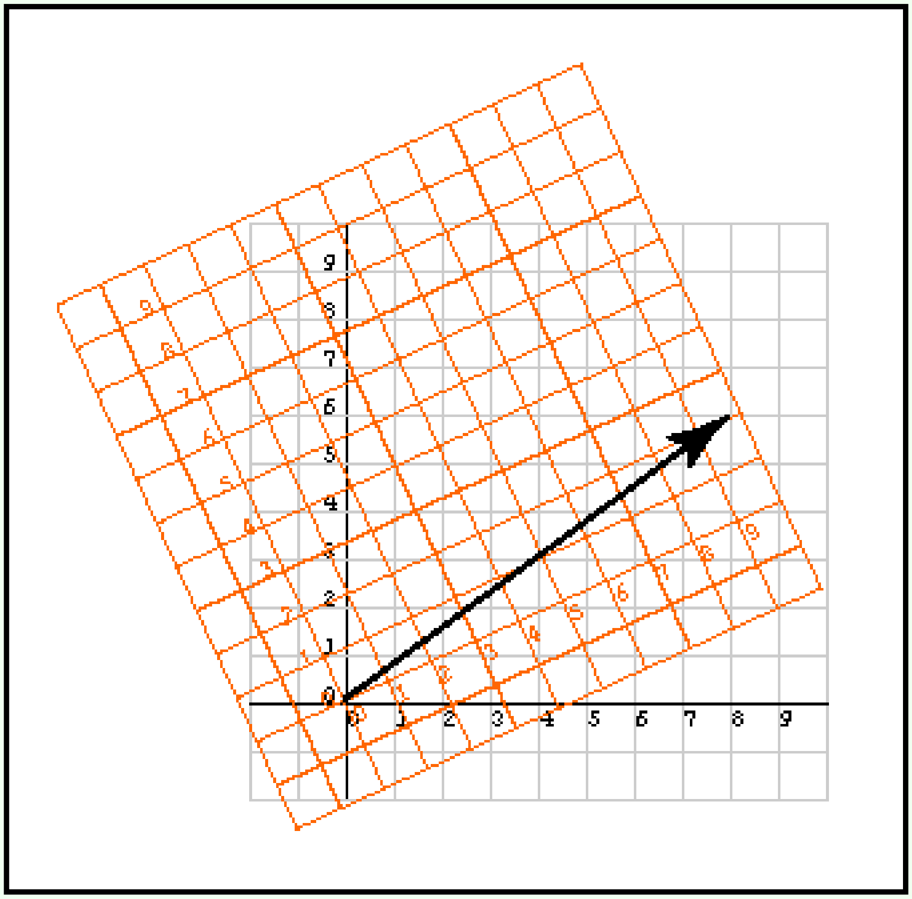
Click Here after you have answered the question

Is the distinction between an object and its representation of any importance in computer science?

A good answer might be:

Yes, it is one of the most important ideas in computer science.

More Practice



It would be good to practice that idea. The diagram shows a vector and two coordinate frames; a light gray frame and an orange frame.

In the light gray frame the vector is represented by $\mathbf{(8, 6)^T}$.

In the orange frame the vector is represented by $\mathbf{(9.8, 2)^T}$.

Say that the axes of both frames are calibrated using the same units (inches, for example).

Using the first representation the length of the vector is:
$\sqrt{(8^2 + 6^2)} = \sqrt{(64 + 36)} = \sqrt{(100)} = 10.0$

Using the second representation the length of the vector is:
$\sqrt{(9.8^2 + 2^2)} = \sqrt{(96 + 4)} = \sqrt{(100)} = 10.0$

QUESTION 10:

If you calculate the length of a vector and get a negative number, what must be true?

Click Here after you have answered the question

A good answer might be:

Your calculator needs new batteries.

Length is always Positive

Since the square of length is a sum of squares, and squares (of real numbers) are always positive, length must always be positive.

$$| \mathbf{a} | = | (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})^\mathbf{T} | = \sqrt{(a_1^2 + a_2^2 + a_3^2)} >= 0$$

The only time the length of a 3D vector is zero is when the vector is the zero vector. In all coordinate frames the 3D zero vector is represented by:

$$\mathbf{0} = (0, 0, 0)^T$$

So its length is:

$$| \mathbf{0} | = \sqrt{(0^2 + 0^2 + 0^2)} = 0$$

Of course, the length of the 2D zero vector is also zero, and it is the only 2D vector with zero length.

QUESTION 11:

Thought questions:

- Will two vectors that are equal to each other have the same length?
- Will two vectors that have the same length always be equal to each other?

Click Here after you have answered the question

A good answer might be:

- Will two vectors that are equal to each other have the same length?
  - --- YES, since corresponding elements must be equal, so corresponding squares must be equal, so the sum must be equal, so the length must be equal.
- Will two vectors that have the same length always be equal to each other?
  - --- NO. It is possible for the sum of the squared elements to be equal without the elements themselves being equal.

Length of the Negative of a Vector

Remember the negative of a vector. If **v** is a vector, then **-v** is a vector pointing in the opposite direction.

If **v** is represented by $\mathbf{(a, b, c)^T}$ then **-v** is represented by $\mathbf{(-a, -b, -c)^T}$.

QUESTION 12:

What is the relation between the length of **v** and the length of **-v**?

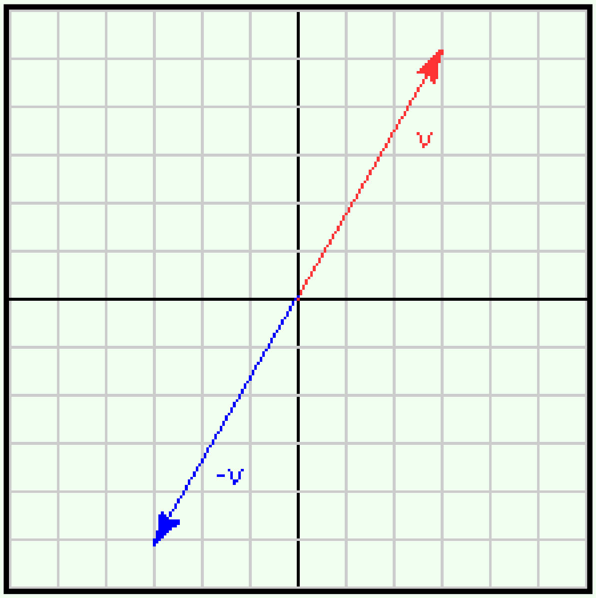
Click Here after you have answered the question

What is the relation between the length of **v** and the length of **-v**?

A good answer might be:

| v | = **| -v |**

Opposite Direction



You can see this mechanically:

| v | = ||(a, b, c)^T|| = $\sqrt{(a^2 + b^2 + c^2)}$

| -v | = ||(-a, -b, -c)^T|| =
$\sqrt{(-a^2 + -b^2 + -c^2)}$ =
$\sqrt{(a^2 + b^2 + c^2)}$

The diagram shows this graphically (in two dimensions).

You would like to say that the two vectors are the same length, but point in opposite directions. In fact, you *can* say that. The next chapter will give you the authority to do so.

QUESTION 13:

What is the length of the vector represented by **(1, -1, 1)^T**?

Click Here after you have answered the question

What is the length of the vector represented by **(1, -1, 1)^T**?

A good answer might be:

$\sqrt{(1^2 + -1^2 + 1^2)} = \sqrt{3}$

End of this Lengthy Chapter

Here are some terms and formulae you may wish to review at length:

- Length of a 2D vector parallel with a coordinate axis.
- Pythagorean Formula.
- Formula for 2D vector length (first version.)
- Formula for 2D vector length (second version.)
- The triangle inequality.
- Formula for 3D vector length.
- Length is always positive or zero.
- Length of the negative of a vector.

Click on a term to see where it was discussed in this chapter. Remember to click on the "Back" arrow of your browser to get back to this page. The next chapter will discuss the *direction* of a vector.

Click here to go back to the main menu.

You have reached the End.

created 08/26/97; revised 11/11/00

Vector Lessons Chapter 5 -- Vector Direction

This chapter discusses a second important vector property: **direction**. The previous chapter discussed the **length** of a vector. These are the two properties that define a vector.

- Orientation of 2D vectors (represented in a particular coordinate frame).
- Ambiguity when using arc tan to calculate angle.
- Converting length and orientation into components of a column matrix.
- Radians and degrees.
- C and Java programing with arc tangent.
-
-

QUESTION 1:

What do you suppose is the direction of the vector represented by:

(4, 0)^T

Click Here after you have answered the question

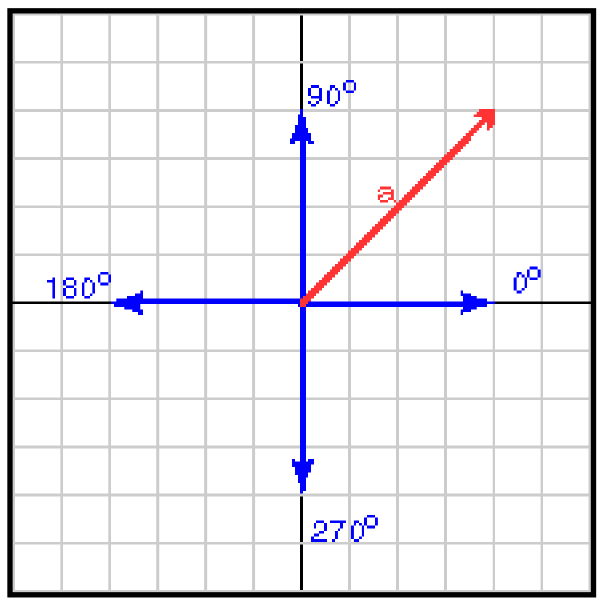
What do you suppose is the direction of the vector represented by:

(4, 0)^T

A good answer might be:

The vector is parallel to the X axis of the coordinate frame we are using. Often this is horizontal, pointing right. (Or, you might have said that the vector is oriented at 0 degrees.)

Vectors Aligned with X-Y Axes



A 2D vector that is aligned with a coordinate axis (as with **(4,0)^T**) has an orientation that is easy to see. It is 0 degrees, 90 degrees, 180 degrees, or 270 degrees.

The orientation of a vector is usually expressed as an angle to the positive x axis of a particular coordinate frame. There are two ways of doing this:

- The angle is 0º to 360º measured as a counter-clockwise rotation from the positive x axis.
- The angle is 0º to +180º measured as a counter-clockwise rotation from the positive x axis, or is 0º to -180º measured as a clockwise rotation from the positive x axis,

QUESTION 2:

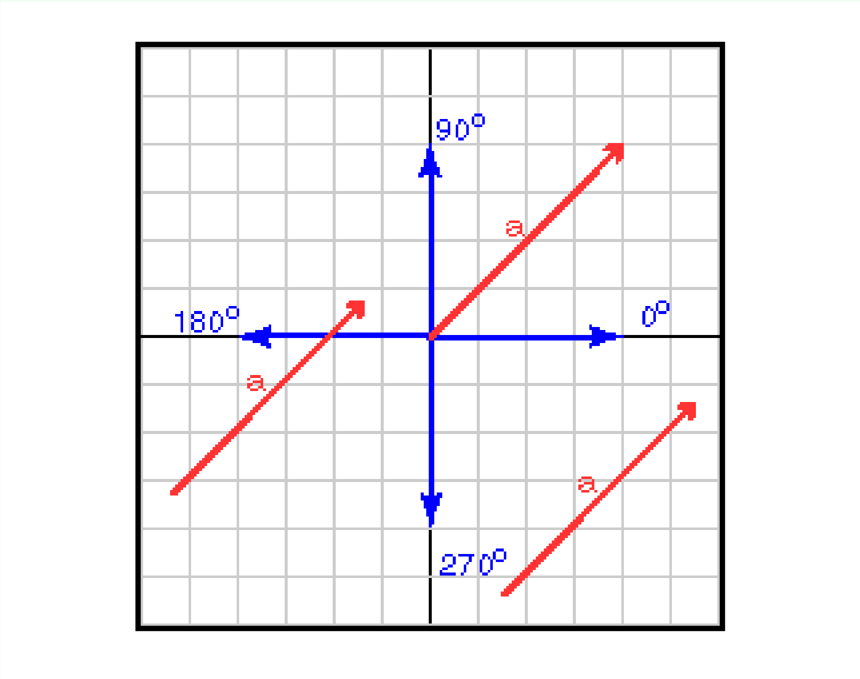
What is the direction of **a**?

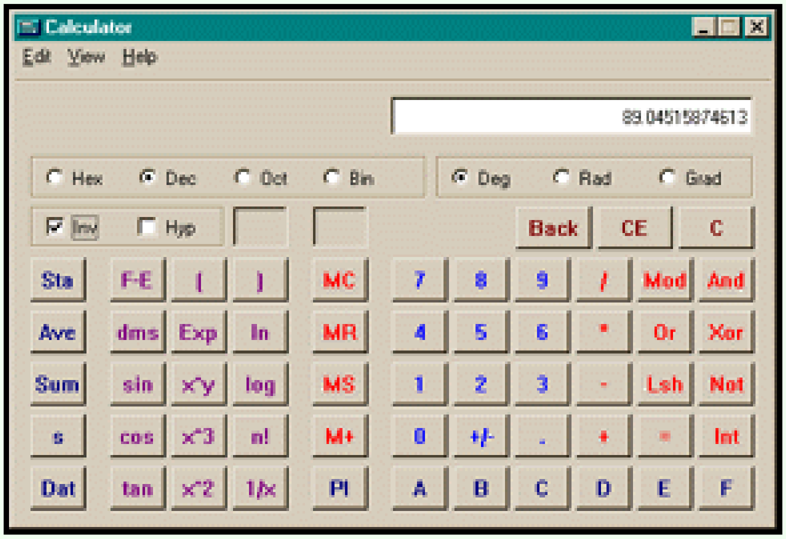
Click Here after you have answered the question

A good answer might be:

a = (4,4)^T is oriented at 45º

Angle from the X Axis





Math should give the same answer: The slope of **a** is

(change in y)/(change in x) = 4.0/4.0 = 1.0.

So the angle is arc tan(1.0) = 45º.

Vectors have no location, so neither the length nor the direction depend on where you draw a vector. The formula for the direction of a 2D vector is

angle of (x, y)^T = arc tan(y/x)

Arc tan(z) means "find the angle that has a tangent of z." When you do this with a calculator, remember that the answer may be in radians or in degrees. Most calculators use both formats, depending on what options you have chosen.

Also, since calculators usually give the answer as -90.0 to +90.0 degrees (or -pi to +pi radians) *you may have to adjust the answer*. More on this later.

You can use the calculator on your computer. If you are using MS Windows, click "scientific" in the "view" menu for a calculator that has arc tan. To calculate arc tan of the number in the display, click the check box labeled "inv", then click "tan."

QUESTION 3:

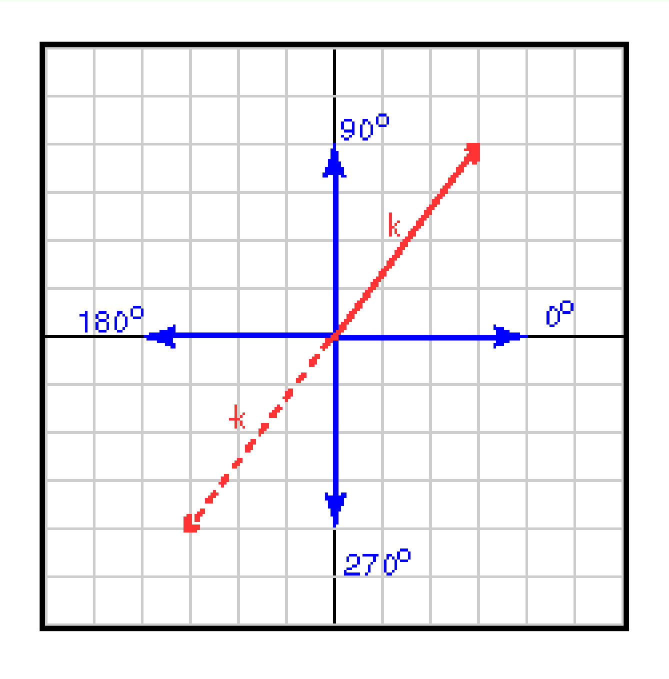
What is the orientation of the vector represented by: $\mathbf{k} = (3,4)^T$?

Click Here after you have answered the question

A good answer might be:

arc tan(y/x) = arc tan(4/3) = arc tan(1.333333333333) = 53.13º

Watch those "-" Signs



The diagram shows the vector represented by **k = (3,4)^T**. Its orientation was calculated to be 53.13º to the positive x axis. That looks about right. The formula worked!

Now calculate the orientation of **-k = (-3,-4)^T**. Plugging into the formula:

```
arc tan( y/x ) = arc tan( -4/-3 ) = ar
               = arc tan(  1.33333333
               =  53.13º
```

Hmm... something is wrong. The formula gave us the same angle for a vector pointing in the opposite direction to the first. The problem is that information is lost when -4 is divided by -3. We can't tell the result from +4 divided by +3. *The formula is not enough to give you the answer;* you should sketch the vector and ajust the answer.

Look at the pictue to see that the orientation of **-k** (expressed in degrees 0--360 counter clockwise from the x axis) is 180º + 53.13º = 233.13º.

QUESTION 4:

What is the orientation of the vector represented by: **p = (3,-4)^T** ? Use the calculator application on your computer.

Click Here after you have answered the question

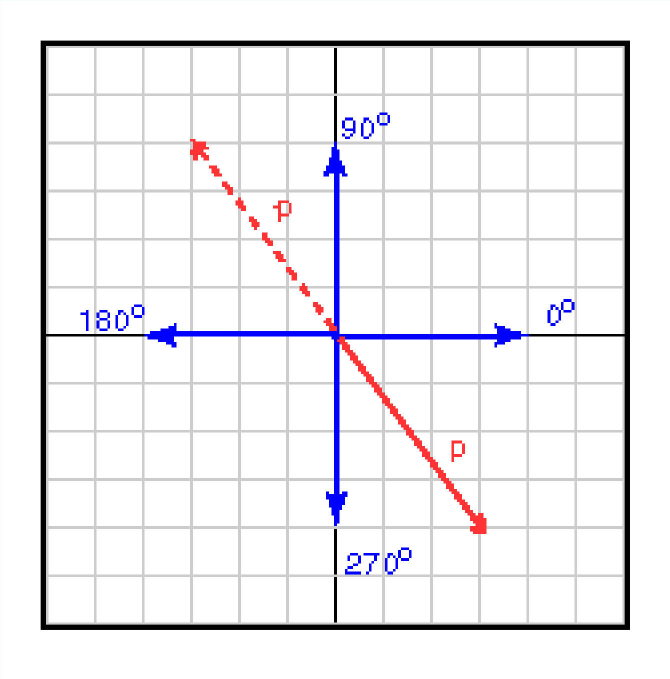
What is the orientation of: $\mathbf{p} = (3,-4)^T$?

A good answer might be:

Plugging the numbers into the MS Windows calculator:

arc tan(y/x) = arc tan(-4/3) = arc tan(-1.333333333333) = -53.13º

More Meddlesome Minuses



The calculation looks about right, in the diagram. If you want the angle to be expressed as a counter-clockwise rotation, then it is 360º -53.130º = 306.870º

Sadly, it looks like the vector **-p** will give you the same result. So what to do? *Always sketch the vector when you are computing its orientation.* Then you can see that the angle you want for **-p** is 180º - 53.130º = 126.87.

QUESTION 5:

What is the orientation of $\mathbf{u} = (-4, -2)^T$? (Sketch the vector first, then use a calculator.)

Click Here after you have answered the question

What is the orientation of $\mathbf{u} = (-4, -2)^T$?

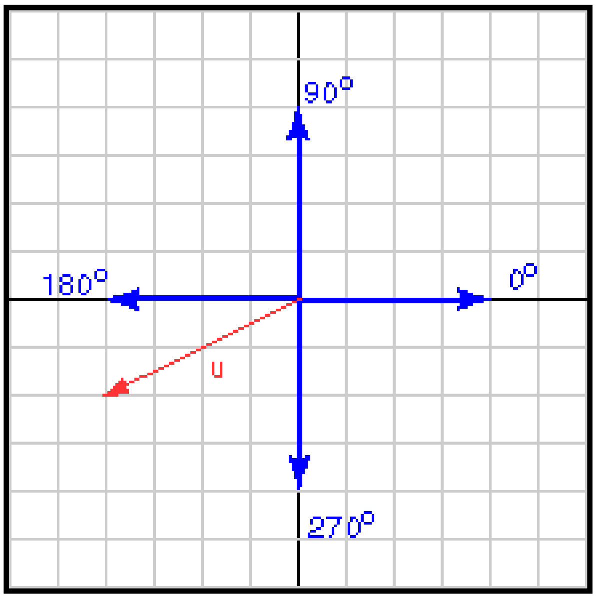
A good answer might be:

By plugging into the formula (and using inv tan of the MS Win calculator):

$$\arctan(y/x) = \arctan(-2/-4) = \arctan(0.5) = 26.565 \text{ degrees}$$

...But from the sketch, this is in the wrong quadrent. If you study the sketch and use geometry, you realize that the complete answer must be $(180 + 26.565)^o = 206.565^o$.

Calculating both Attributes



Say that (using a particular coordinate frame) you know that a vector is represented by the column matrix $\mathbf{(x, y)^T}$.

You can calculate the two attributes of the vector:

orientation of $(x, y)^T = \arctan(y/x)$

$| \mathbf{(x, y)^T} | = \sqrt{(x^2 + y^2)}$

In a few pages we will start with length and orientation and calculate the x and y for a particular coordinate frame.

QUESTION 6:

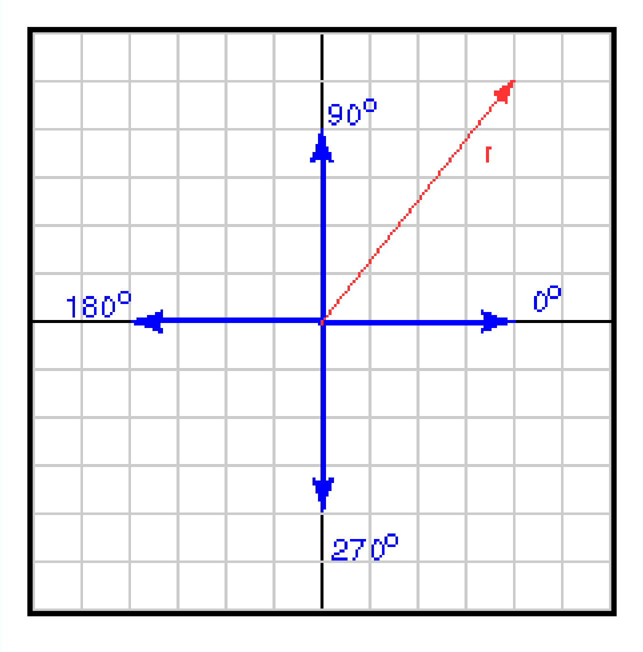
Sketch the vector **r** represented by $\mathbf{(4, 5)^T}$. Make an eyeball estimate of its length and direction. Then do the math to get the exact answer.

Click Here after you have answered the question

A good answer might be:

- **| r | = | (4,5)^T |** = $\sqrt{(16+25)}$ = 6.40
- direction = arc tan{ 5/4 } = 51.34º

Length and Direction converted to (x, y)



Say that you have been given a length and a direction and want to express them as a 2D column vector.

1. Draw a sketch.
2. Calculate x by projecting the length onto the x-axis (usually length*cos(θ))
3. Calculate y by projecting the length onto the y-axis (usually length*sin(θ))
4. Check answers against the sketch.

Of course, you can't wait to practice this (it brings back those fond memories of trigonometry.)

QUESTION 7:

A vector has length 4 and orientation 150º. Express the vector as a column vector.

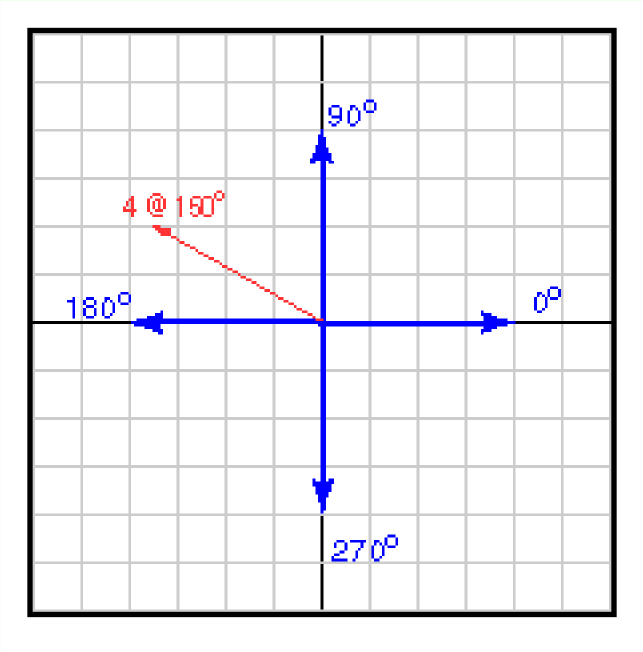
Click Here after you have answered the question

A vector has length 4 and orientation 150º. Express the vector as a column vector.

A good answer might be:

The 2D vector is **(-3.464, 2.0)^T**

Radians



The steps in this calculation are:

1. Draw a sketch: see diagram
2. Calculate x by projecting the length onto the x-axis:
 4 * cos(150) = -3.464
3. Calculate y by projecting the length onto the y-axis:
 4 * sin(150) = 2.0
4. Check answers against the sketch: Looks OK.

Be cautious about plugging into these formulae and expecting correct answers, especially when programming in C or Java. Math libraries for a programming language can do unexpected things if you are not careful.

There are three places to be especially cautious:

1. The argument for sin(), cos(), tan() is expected in *radians*. The return value of atan() is in radians.
2. The argument for most math functions is expected to be a *double*. If you supply a float or an int, you won't get a error message, just a horribly incorrect answer.
3. There are several versions of "arc tan" in most C libraries, each for a different range of output vaues.

Now would be a good time think about *radians*. Usually in professional circles, angles are expressed in radians. Angles are measured counterclockwise from the positive x axis (or sometimes a negative angle is measured clockwise from the positive x axis.

There are 2 pi radians per full circle. Or 2 pi radians = 360º

QUESTION 8:

Fill in the blanks so that vector[0] gets the x component and vector[1] gets the y component of the vector.

```
#include math.h
#define PI 3.14159265

double length, angle;
double vector[2];
 . . .
length = some value
angle  = some number of degrees

vector[0] =  ________________

vector[1] =  ________________
```

(If you don't know C, just pretend it is Java.)

Click Here after you have answered the question

A good answer might be:

```
#include math.h
#define PI 3.14159265

double length, angle;
double vector[2];
 . . .
length = some value
angle = some number of degrees

vector[0] =  length * cos( angle*PI/180.0 )

vector[1] =  length * sin( angle*PI/180.0 )
```

(Unfortunately, different compilers use different symbols for PI, and define it it different header files.)

Practice with Radians

Recall the steps to take in converting a vector given as length and direction into **(x,y)**:

1. Draw a sketch.
2. Calculate x by projecting the length onto the x-axis
   (usually length*cos(θ))
3. Calculate y by projecting the length onto the y-axis
   (usually length*sin(θ))
4. Check answers against the sketch.

The steps are the same (of course) if the angle is given in radians.

QUESTION 9:

A vector is 4.5 units long oriented at 0.70 radians. Express the vector as $\mathbf{(x, y)^T}$.

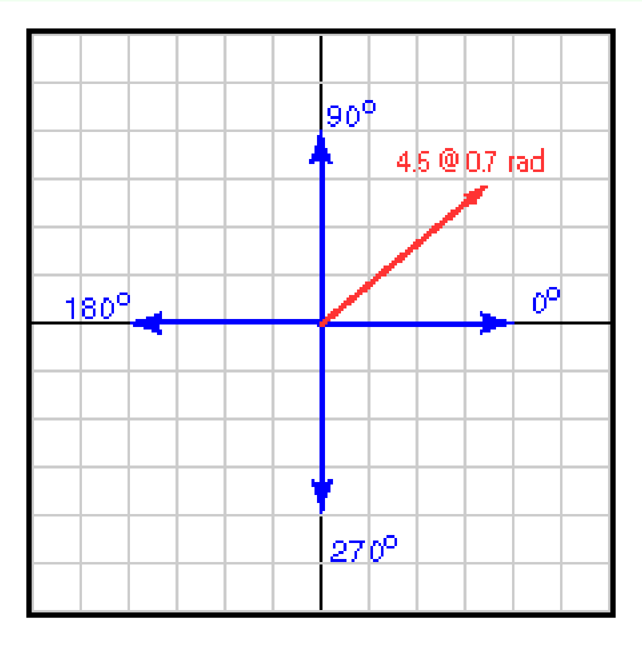
Click Here after you have answered the question

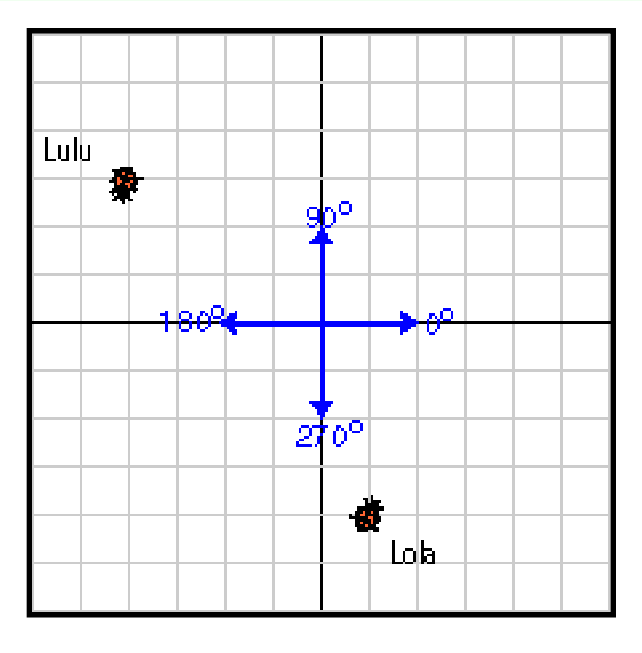
A vector is 4.5 units long at 0.70 radians. Express the vector as **(x, y)^T**.

A good answer might be:

1. Draw a sketch. See Below.
2. Calculate x by projecting the length onto the x-axis:
   4.5*cos(0.70) = 3.442
3. Calculate y by projecting the length onto the y-axis
   4.5*sin(0.70) = 2.899
4. Check answers against the sketch.

Lost Bugs





Your sketch should look something like the one at the left. Now it is time for another story problem.

Lola the Ladybug is lost. Luckily, Lola has a cellular phone and a compass. Lola is located at the point **(1, -4)**. Lulu is located at the point **(-4, 3)**,

QUESTION 10:

Call up Lola and tell her in what direction and for what distance she should walk to reach her friend Lulu. (Usually a compass regards North as zero degrees; ignore this and use our usual method.)

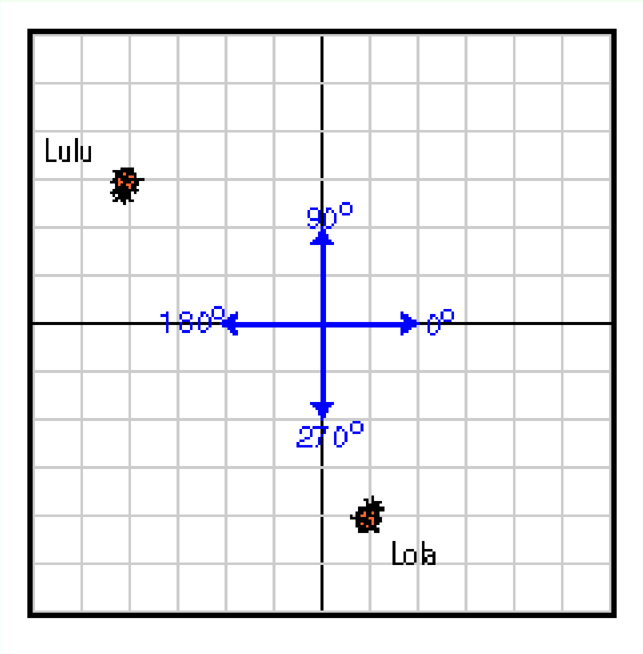
Click Here after you have answered the question

- Lola is located at **(1, -4)**.
- Lulu is located at **(-4, 3)**,

A good answer might be:

Direction = 125.54º, distance = 8.60.

Buggy Directions



To get the answer you had to recall how to calculate displacement. To calculate the displacement vector between two points (or bugs):

(where you want to be) - (where you are) = (displacement you need)

So **(-4, 3) - (1, -4)** = **(-5, 7)**.

But Lola needs distance and heading:

- distance = $\sqrt{(25 + 49)}$ = 8.60
- direction = arc tan(7/-5) = arc tan(-1.4) = -54.46º
- (looking at the diagram to adjust the direction):

direction = 180 - 54.46 = 125.54º

QUESTION 11:

Will the ideas about 2D vector orientation presented in this chapter work for 3D space?

Click Here after you have answered the question

A good answer might be:

Not easily. Orientation of 3D vectors will take more work, and another chapter.

End of this Lesson

Here are some terms and formulae you may wish to review before you get lost:

- Orientation of a 2D vector aligned with X or Y axis.
- Formula that nearly gives you the orientation of a vector.
- Dealing with negative components in the column matrix.
- Converting length and direction to an ordered pair.
- Relation between radians and degrees.

Click on a term to see where it was discussed in this chapter. Remember to click on the "Back" arrow of your browser to get back to this page. The next chapter will discuss the product of a vector and a scalar. This will be useful in several important applications, such as in quizzes and midterms.

Click here to go back to the main menu.

You have reached the End.

created 08/26/97; revised 08/12/00

CHAPTER 6 -- Scaling and Unit Vectors

This chapter discusses how to multiply a vector by a real number (called **scaling**), and how this is used to construct **unit vectors**. Unit vectors are used to show direction in 3D space, and are essential for 3D graphics.

- Scaling.
- Unit vectors.
- Normalizing a vector.
- Variables in vector equations.

QUESTION 1:

Change the elements of this column matrix: $\mathbf{(3, 4)^T}$ so that the vector it represents is twice as long and remains pointing in the same direction. (Officially you don't know how to do this. Take a guess.)

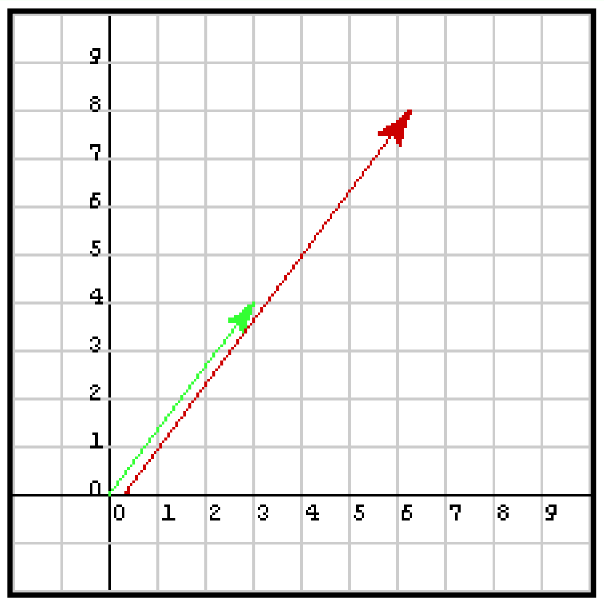
Click Here after you have answered the question

Change the elements of this column matrix: $(3, 4)^T$ so that the vector it represents is twice as long and remains pointing in the same direction.

A good answer might be:

You could carefully work out the correct answer using what you already know. Or you could take a guess and see if it works: 2 times $(3, 4)^T$ = $(6, 8)^T$

Scaling



Does the guess work?

- $(\text{New length})^2 = 6^2 + 8^2 = 2^2 3^2 + 2^2 4^2 =$

$2^2(3^2+4^2) = 2^2(\text{old length})^2$
- So, taking roots, (New length) = 2(old length)
- By similar triangles, the direction is the same.

Or you could notice: new direction = arc tan(8/6) = arc tan(4/3) = old direction. It looks like the guess worked.

In talking about vectors, a real number is sometimes called a *scalar*.

Scaling a geometrical vector means keeping its orientation the same but changing its length by a scale factor. It is like changing the scale of a picture; the distances expand or shrink, but the directions remain the same.

If a vector is represented by a column matrix $(x, y)^T$ then scaling by the a number multiplies each element:

$$a(x, y)^T = (ax, ay)^T$$

This works for 3 dimensional vectors also:

If **v** is represented by $\mathbf{(x, y, z)^T}$ then a**v** is represented by $\mathbf{(ax, ay, az)^T}$

QUESTION 2:

What is $0.5*(36.4, -18.9)^T$?

Click Here after you have answered the question

What is 0.5 $(36.4, -18.9)^T$?

A good answer might be:

$(0.5*36.4, 0.5* -18.9)^T = (18.2, -9.45)^T$

A few more fun Facts

There are several properties of scaling that you can probably guess:

QUESTION 3:

If **v** is a vector, what is:

- -1**v**
- 0**v**

Click Here after you have answered the question

A good answer might be:

- $-1\mathbf{v} = \mathbf{-v}$
- $0\mathbf{v} = \mathbf{0}$

Answer is always a Vector

Notice that even though the above two answers are easy to guess, there is some subtlety to them:

- $-1\mathbf{v} = \mathbf{-v}$ real number * vector = vector
- $0\mathbf{v} = \mathbf{0}$ real number * vector = vector

The output of scaling is a vector. The boldface **0** in the last equation is the vector represented by $(0,0,0)^T$, not the real number zero.

QUESTION 4:

What is:

$2 + 4(3, 2, 5)^T$

Click Here after you have answered the question

What is:

$$2 + 4(3, 2, 5)^T$$

A good answer might be:

(Trick Question!) The expression is ill-formed. There is no operator + that does this: real number + vector

Expressions with Mixed Types

It is helpful to use parentheses to show exactly what you mean in expressions containing mixed types and operators. Look at this:

$$(2 + 1)(3, -5)^T + 4(1, 2)^T$$

The first "+" is scalar addition; the second "+" is column matrix addition. You should perform the operations like this:

$$(2 + 1)(3, -5)^T + 4(1, 2)^T = 3(3, -5)^T + 4(1, 2)^T = (3*3, 3* -5)^T + (4*1, 4*2)^T$$
$$= (9, -15)^T + (4, 8)^T = (13, -7)^T$$

This is not as bad as it looks, as long as you keep track of what operators you have available, what types of operands they take, and what their results are.

QUESTION 5:

Evaluate the following:

$$-1\ (1, 2, -3)^T - (2 + 1)(1, 0, 1)^T$$

Click Here after you have answered the question

A good answer might be:

$-1\ (1, 2, -3)^T - (2 + 1)(1, 0, 1)^T = (-1, -2, 3)^T - 3(1, 0, 1)^T$
$= (-1, -2, 3)^T + (-3, 0, -3)^T = (-4, -2, 0)^T$

More Algebra!

If there is an unknown or two in the mix, you can use algebra in the usual way to solve for them. Just be careful that each move you make results in a valid expression. For example, solve the following for a and y:

$$a(1, y)^T + 2(0, 5)^T = (2, 20)^T$$

You might say that this cannot be done, since there are two unknowns (a and y) but only one equation. However:

$$a(1, y)^T + 2(0, 5)^T = (2, 20)^T$$

$$(a, ay)^T + (0, 10)^T = (2, 20)^T$$

$$(a, ay+10)^T = (2, 20)$$

Corresponding elements must be equal. So:

```
a       =  2
ay + 10 = 20
2y      = 10
y       =  5
```

This last manuever is a very common trick, called *equating corresponding elements*.

QUESTION 6:

Here is an opportunity to practice. Find x and y :

$$4(\ -1,\ y)^T \ + \ 2(\ 3x,\ 10)^T \ = (8,\ 24)^T$$

Click Here after you have answered the question

A good answer might be:

$$4(-1, y)^T + 2(3x, 10)^T = (8, 24)^T$$
$$(-4, 4y)^T + (6x, 20)^T = (8, 24)^T$$
$$(6x-4, 4y+20)^T = (8, 24)^T$$

$$6x-4 = 8$$
$$4y+20 = 24$$

$$6x = 12;\ x = 2$$
$$4y = 4;\ y = 1$$

Equating Corresponding Elements

In the original equation (and in the two re-writes of it) the "=" sign means "vector equality." Equating corresponding elements yields two equations where the "=" sign means "scalar equality" (or "real number equality.")

The "trick" of equating corresponding elements converts a column matrix equation into several scalar equations, one equation for each dimension of the column matrix. Then, sometimes, the scalar equations can be solved for the unknowns. But this does not always work.

Here is a case where it does not work: Find x and y :

$$4(x, 5)^T + 2(y, 1)^T = 2(12, 11)^T$$

This looks OK, but doing some work brings up a problem:

$$4(x, 5)^T + 2(y, 1)^T = 2(12, 11)^T$$
$$(4x, 20)^T + (2y, 2)^T = (24, 22)^T$$
$$(4x+2y, 22)^T = (24, 22)^T$$

$$4x + 2y = 24$$
$$22 = 22$$

The two scalar equations we get by equating corresponding elements do not contain enough information to proceed.

QUESTION 7:

Find a and x :

$$a(-1, 5)^T + 2(3x, 10)^T = (8, 25)^T$$

(Before you start figuring take a guess: can it be done?)

Click Here after you have answered the question

A good answer might be:

Can it be done? Yes, but you might be worried, at first.

$$a(-1, 5)^T + 2(3x, 10)^T = (8, 25)^T$$
$$(-a, 5a)^T + (6x, 20)^T = (8, 25)^T$$
$$(6x-a, 5a+20)^T = (8, 25)^T$$

Equating corresponding elements:

$$6x-a = 8$$
$$5a+20 = 25; \quad 5a = 5; \quad a = 1$$

$$6x-1 = 8; \quad 6x = 9; \quad x = 9/6 = 3/2$$

Scaling with values less than One

In the definition of scaling:

$$a(x, y)^T = (ax, ay)^T$$

it is OK if "a" is smaller than one. For example:

$$0.5(x, y)^T = (0.5*x, 0.5*y)^T$$

Sometimes this is written as:

$$\frac{(x, y)}{2} = (x/2, y/2)$$

QUESTION 8:

(Review:) What is the length of $(1,0)^T$?

Click Here after you have answered the question

What is the length of $(1,0)^T$?

A good answer might be:

1.0

Unit Vector

The vector has a length of one because it is aligned with the x axis (in the current coordinate frame). When represented by a column matrix its only non-zero element is 1.0. Of course, other vectors, not aligned with any axis, can have a length of one.

> A **unit vector** is a vector with a length of one.

QUESTION 9:

Does the following represent a unit vector: $(1, 1, 1)^T$?

Click Here after you have answered the question

Does the following represent a unit vector: $(1, 1, 1)^T$?

A good answer might be:

No, its length is $\sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3} = 1.7320508$

Creating Unit Vectors

Unit vectors have a length of one. If you have a particular vector **v** you can use it to make a unit vector. This is called **normalizing** the vector:

1. Calculate the length of **v**, $|\mathbf{v}|$.
2. Scale **v** by the inverse of its length: $\mathbf{v} / |\mathbf{v}|$

Often this idea is written as a formula (the little subscript "u" is supposed to mean "unit vector"):

$$\mathbf{v_u} = \mathbf{v} / |\mathbf{v}|$$

If $\mathbf{v} = (x, y, z)^T$ then:

$$\mathbf{v_u} = \mathbf{v} / |\mathbf{v}| = (x / |\mathbf{v}|, \; y / |\mathbf{v}|, \; z / |\mathbf{v}|)^T$$

QUESTION 10:

Normalize $(3, 4)^T$.

Click Here after you have answered the question

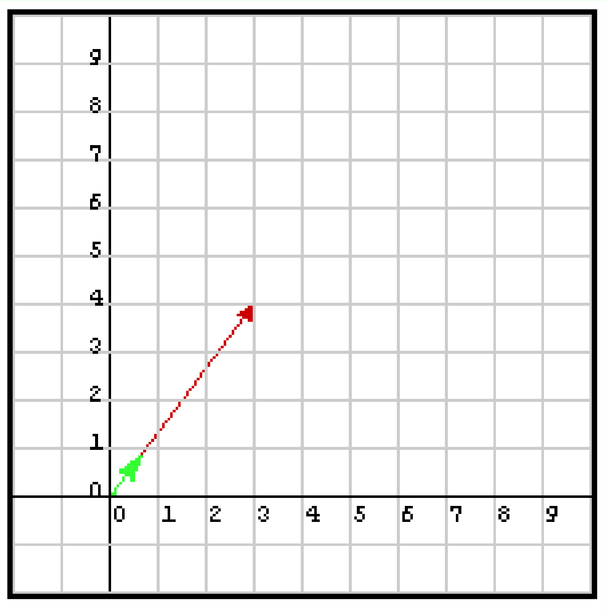
Normalize $(3, 4)^T$.

A good answer might be:

The is the hypotenuse of a 3-4-5 right triangle, so its length is 5. The unit vector is:

$$1/5\ (3,4)^T \quad = \quad (3/5,\ 4/5)^T \quad = \quad (0.6,\ 0.8)^T$$

Direction of a Unit Vector



Scaling changes the length of a vector but not its direction. If $\mathbf{v_u}$ is the unit vector corresponding to **v**, then $\mathbf{v_u}$ and **v** have the same orientation.

This sounds plausible, but a demonstration might not hurt:

1. Start with $\mathbf{v} = (3, 4)^T$ as above.
2. Form $\mathbf{v_u} = (3/5, 4/5)^T$.
3. The direction of **v** is arc tan(4/3).
4. The direction of $\mathbf{v_u}$ is arc tan((4/5) / (3/5)) = arc tan((4/5) * (5/3)) = arc tan(4/3).

QUESTION 11:

Construct a unit vector in the same direction as $\mathbf{w} = (4, 6)^T$.

Click Here after you have answered the question

Construct a unit vector in the same direction as $\mathbf{w} = (4, 6)^T$.

1. $|\mathbf{w}| = \sqrt{(16 + 36)} = \sqrt{52}$
2. $\mathbf{w_u} = (4, 6)^T / (\sqrt{52})$

Check on the Answer

Well, that was fun. Now check that the answer is correct:

$$|\mathbf{w_u}|^2 = (4^2 + 6^2) / (\sqrt{52})^2$$
$$|\mathbf{w_u}|^2 = (16 + 36)/52 = 52/52 = 1.0$$
$$|\mathbf{w_u}| = 1.0$$

A unit vector formed from a three dimensional vector will point in the same direction as the original.

QUESTION 12:

Normalize: $\mathbf{g} = (-3, 4, -1)^T$

Click Here after you have answered the question

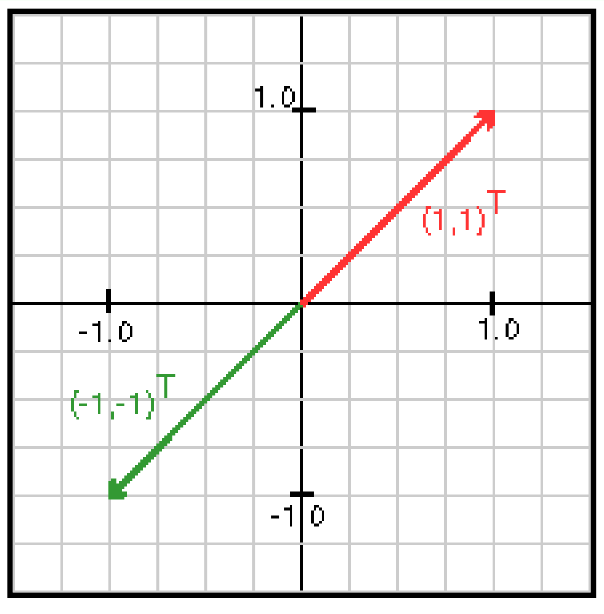
Normalize: **g** = $(-3, 4, -1)^T$

A good answer might be:

$| \mathbf{g} | = \sqrt{(9 + 16 + 1)} = \sqrt{26}$

$\mathbf{g_u} = (-3, 4, -1)^T / (\sqrt{26})$

Opposite Direction



This represents a unit vector oriented at 45°:

$(1, 1)^T / (\sqrt{2})$.

The length is $\sqrt{(1^2/2 + 1^2/2)} = 1$. The angle is arctan(1/1) = 45°. How would you make a unit vector that is pointed in the opposite direction? Negate each component:

$-(1, 1)^T / (\sqrt{2}) = (-1, -1)^T / (\sqrt{2})$.

QUESTION 13:

Create a unit vector that points in the opposite direction as $(3, 0, 2)^T$.

Click Here after you have answered the question

Create a unit vector that points in the opposite direction as $(3, 0, 2)^T$.

A good answer might be:

Length of the vector = $\sqrt{(9+4)}$ = $\sqrt{13}$
Unit vector in the first direction = $(3, 0, 2)^T / (\sqrt{13})$
Unit vector in the opposite direction = $(-3, 0, -2)^T / (\sqrt{13})$

End of this Chapter

You should scale down your plans for a party weekend and study these topics instead:

- Definition of scaling.
- Definition of a scalar.
- Some properties of scaling.
- Equating corresponding elements in vector algebra.
- Unit Vectors.
- Normalizing a vector.
- Orientation of a unit vector.

Click on a term to see where it was discussed in this chapter. Remember to click on the "Back" arrow of your browser to get back to this page. The next chapter will discuss the *dot product* of two vectors.

Click here to go back to the main menu.

You have reached the End.

created 08/26/97; revised 08/12/00

CHAPTER 7 -- Dot Product

Examine the operations that have been discussed so far:

1. Vector addition: vector + vector = vector
2. Vector subtraction: vector - vector = vector
3. Difference of points: point - point = vector
4. Scaling: real * vector = vector

This chapter discusses the **dot product**, which takes two vectors as operands and produces a real number as its output. Sometimes the dot product is called the *inner product*. Sometimes the dot product is called the *scalar product*, which should not be confused with the operation called *scaling* (but probably will be).

QUESTION 1:

Why is the dot product sometimes called the scalar product?

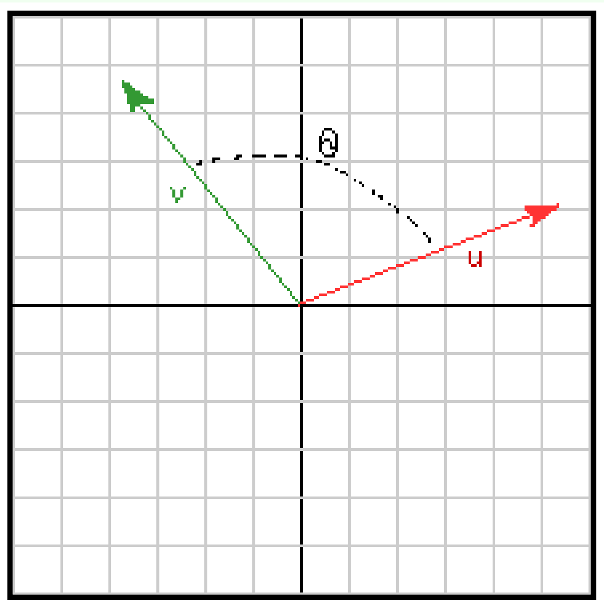
Click Here after you have answered the question

Why is the dot product sometimes called the scalar product?

A good answer might be:

Because it takes two vectors and produces a *scalar* (a real number.)

The Dot Product



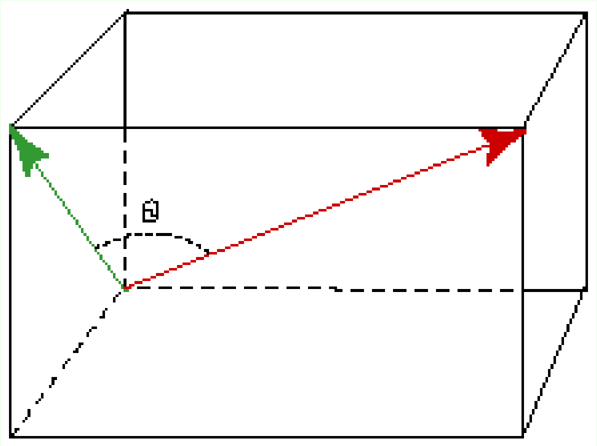
For geometrical vectors in two or three dimensional space, the **dot product** of two vectors u and v is:

$$u \cdot v \;=\; |u|\,|v| \cos\theta$$

θ is the angle between the two vectors.

The dot product is indicated by the dot, " · " between the two vectors. Don't write two vectors next to each other like this: **uv** when you want the dot product. Always put a dot between them: **u · v** .

In 2D the two vectors lie in a plane (of course) and the angle between them is easy to visualize.



In 3D two vectors also lie in a plane embedded within the 3D space, except when the two vectors are co-linear (when they both point in the same direction).

When two vectors are co-linear, the angle between them is zero and so:

$$u \cdot v \;=\; |u|\,|v| \cos 0 \;=\; |u|\,|v|\, 1 \;=\; |u|\,|v|$$

QUESTION 2:

If $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, what is $\mathbf{v} \cdot \mathbf{u}$?

Click Here after you have answered the question

If $u \cdot v = |u|\,|v| \cos\theta$, what is $v \cdot u$?

A good answer might be:

$v \cdot u = |v|\,|u| \cos\theta = |u|\,|v| \cos\theta = u \cdot v$

Commutative

The dot product is **commutative** The order of operands does not make any difference.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$

Another property is:

$$\mathbf{0} \cdot \mathbf{0} = 0.$$

This means that the dot product of the *zero vector* with itself results in the *scalar value* of zero. There are two different kinds of zero in the equation. Remember that operands of the dot product are two vectors, and the output is a scalar (a real number).

QUESTION 3:

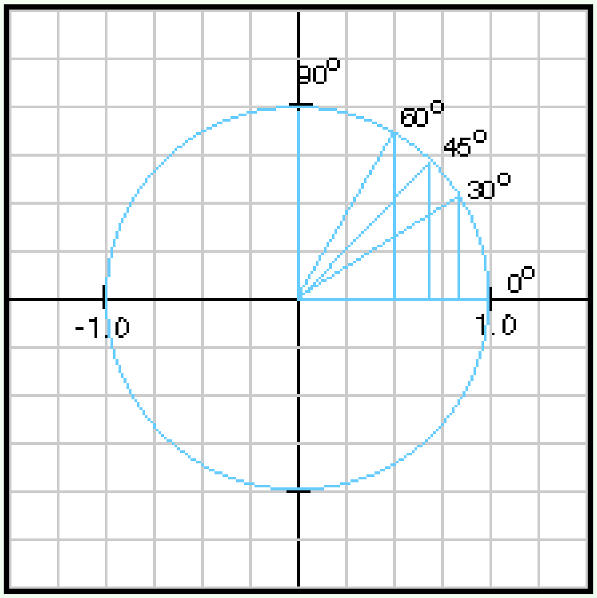
What is $a\,(\mathbf{u} \cdot \mathbf{v})$ where "a" is a scalar?

Click Here after you have answered the question

A good answer might be:

a (**u** · **v**) = a |u| |v| cos θ = | au | |v| cos θ = (a**u**) · **v**

Cosine of 90°



You may be somewhat fuzzy about how the cosine function behaves. Rather than memorize abstract stuff, I prefer to visualize the unit circle with its radius projected onto the x-axis.

From the picture, the cosine of 0° = 1.0, the cosine of 30° = 0.866, the cosine of 45° = 0.707, the cosine of 60° = 0.500, the cosine of 90° = 0.0.

Recall that

u · v = |u| |v| cos θ

QUESTION 4:

Two vectors are oriented at 90° to each other. What is their dot product?

Click Here after you have answered the question

Two vectors are oriented at 90° to each other. What is their dot product?

A good answer might be:

If u and v are orthogonal, then $u \cdot v = |u|\,|v| \cos 90° = |u|\,|v|\, 0.0 = 0.0$

Dot Product of Orthogonal Vectors

This fact is of fundamental importance. It works for vectors of all dimensions:

> The dot product of orthogonal vectors is zero.

"Orthogonal" means "oriented at 90° to each other". To keep things consistent, the zero vector is regarded as orthogonal to all other vectors since $\mathbf{0} \cdot \mathbf{v} = 0.0$ for all vectors **v**.

QUESTION 5:

Say that two vectors **s** and **t** have a dot product that is zero.

- What can you say about the relative orientation of **s** and **t**?
- What can you say about the lengths of **s** and **t**?

Click Here after you have answered the question

A good answer might be:

Say that two vectors **s** and **t** have a dot product that is zero.

- What can you say about the relative orientation of **s** and **t**?
  s and **t** are orthogonal.
- What can you say about the lengths of **s** and **t**?
  You can't say anything, not from what you know about the dot product, anyway.

Pure Orthogonality Detector

When two vectors are orthogonal (to each other) then their dot product is zero, regardless of their lengths. The dot product "detects" orthogonality no matter what the lengths.

Now look at the dot product of a vector with itself:

$$v \cdot v = |v|\,|v| \cos 0° = |v|\,|v|\,1.0 = |v|^2$$

The dot product of a vector with itself yields the square of its length, or:

$$|\,v\,| = \sqrt{(v \cdot v)}$$

Used in this fashion, the dot product is a *pure length detector*. Since the two properties of a vector are length and orientation, you might suspect that the dot product is useful.

QUESTION 6:

- Will $v \cdot v$ ever be negative?
- Will $v \cdot v$ ever be zero?

Click Here after you have answered the question

A good answer might be:

- Will $v \cdot v$ ever be negative?
  No, since length is always zero or positive.
- Will $v \cdot v$ ever be zero?
  Yes, when v is the zero vector.

Properties of Dot Product

Another property of the dot product is:

$$(a\mathbf{u} + b\mathbf{v}) \cdot \mathbf{w} = (a\mathbf{u}) \cdot \mathbf{w} + (b\mathbf{v}) \cdot \mathbf{w}, \quad \text{where a and b are scalars}$$

Here is the list of properties of the dot product:

1. $u \cdot v = |u|\,|v| \cos\theta$
2. $u \cdot v = v \cdot u$
3. $u \cdot v = 0$ when u and v are orthogonal.
4. $\mathbf{0} \cdot \mathbf{0} = 0$
5. $|\,v\,|^2 = v \cdot v$
6. $a\,(\mathbf{u} \cdot \mathbf{v}) = (a\,\mathbf{u}) \cdot \mathbf{v}$
7. $(a\mathbf{u} + b\mathbf{v}) \cdot \mathbf{w} = (a\mathbf{u}) \cdot \mathbf{w} + (b\mathbf{v}) \cdot \mathbf{w}$

QUESTION 7:

(Trick Question:) What is $u \cdot v \cdot w$?

Click Here after you have answered the question

A good answer might be:

u · v · w makes no sense. If u · v = a (a scalar), then a · w makes no sense. If v · w = b (a scalar), then u · b makes no sense. Either way the original expression makes no sense.

Dot Product of Column Matrices

The dot product is also defined for the column matrices that represent two vectors. For two-dimensional column matrices the **dot product** is defined as:

Let $\mathbf{a} = (a_1, a_2)^T$ and let $\mathbf{b} = (b_1, b_2)^T$

Then the dot product is:

$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$

Multiply corresponding elements of each vector, then add up the products. The result is a scalar value.

Sometimes the dot product is written like this: $\mathbf{a}^{\mathbf{T}}\, \mathbf{b}$ (but it is defined the same way). The reason for this second, odd notation will be apparent in a later chapter when matrix multiplication is discussed.

Here is an example:

- $\mathbf{a} = (1, 2)^T$
- $\mathbf{b} = (3, 4)^T$
- then $\mathbf{a} \cdot \mathbf{b} = 1*3 + 2*4 = 3 + 8 = 11$

QUESTION 8:

- $\mathbf{a} = (1, 2)^T$
- $\mathbf{b} = (3, 4)^T$
- What is: $\mathbf{b} \cdot \mathbf{a}$?

Click Here after you have answered the question

A good answer might be:

- **a** = $(1, 2)^T$
- **b** = $(3, 4)^T$
- **b · a** = 3*1 + 4*2 = 1*3 + 2*4 = **a · b**

Same Properties

The dot product between column matrices has the same properties as the dot product between vectors. Here is another example:

1. **p** = $(-2, 5)^T$
2. **q** = $(3, -1)^T$
3. then **p · q** = (-2)*3 + 5*(-1) = -6 + -5 = -11

Here is a more realistic example:

1. **s** = $(1.082, -3.224)^T$
2. **t** = $(2.381, 7.009)^T$
3. then **s · t** = 1.082*2.381 + -3.224*7.009 = 2.576242 + -22.597016 = -20.020774

In all cases the dot product takes two column matrix operands and yields one scalar value.

QUESTION 9:

Form the dot product of: $(-1, 3)^T$ and $(2, 4)^T$.

Form the dot product of: $(2, 4)^T$ and $(-1, 3)^T$.

Click Here after you have answered the question

A good answer might be:

$(-1, 3)^T \cdot (2, 4)^T = -1*2 + 3*4 = -2 + 12 = 10$

$(2, 4)^T \cdot (-1, 3)^T = 2*(-1) + 4*3 = -2 + 12 = 10$

Commutative

The dot product of column matrices is commutative:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

As with the dot product of vectors, the order of operands makes no difference. Write out the definition of dot product for both arrangements of **a** and **b**:

$$\mathbf{a} \cdot \mathbf{b} = (a_1, a_2)^T \cdot (b_1, b_2)^T = a_1b_1 + a_2b_2$$

$$\mathbf{b} \cdot \mathbf{a} = (b_1, b_2)^T \cdot (a_1, a_2)^T = b_1a_1 + b_2a_2 = a_1b_1 + a_2b_2$$

(Just in case your eyes have glazed over: notice that the stuff after the last "=" sign is the same in each case.)

QUESTION 10:

(Review:) What do you think might be true about:

$$\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$$

Click Here after you have answered the question

What do you think might be true about:

$\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$

A good answer might be:

It makes no sense. Remember that the dot product of two vectors (or column matrices) makes a real number. There is no such thing as the dot product between a real number and a vector (or column matrix).

Mixed Types == Trouble

There is no such thing as the dot product between a real number and a column matrix.

$\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c} = \text{real} \cdot \mathbf{c}$ <=== garbage

or:

$\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \text{real}$ <=== garbage

When dealing with dot products, keep track of the types of operands and results, as illustrated by the previous question.

QUESTION 11:

What is $2(-1, 2)^T \cdot (4, 1)^T$?

Click Here after you have answered the question

A good answer might be:

This *does* make sense:

$$2(-1, 2)^T \cdot (4, 1)^T = (-2, 4)^T \cdot (4, 1)^T = -2*4 + 4*1 = -8 + 4 = -4$$

(Notice that there is no "dot" between the 2 and the vector following it, so this means "scaling," not dot product.)

Dot Product in Three Dimensions

The dot product is defined for 3D column matrices. The idea is the same: *multiply corresponding elements of both column matrices, then add up all the products*.

Let $\mathbf{a} = (a_1, a_2, a_3)^T$
Let $\mathbf{b} = (b_1, b_2, b_3)^T$

Then the dot product is:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Both column matrices must have the same number of elements.

$$(1, 2, 3)^T \cdot (6, 7, 8)^T = 1*6 + 2*7 + 3*8 = 44$$
$$(-1, 2, -3)^T \cdot (1, -2, 3)^T = (-1)(1) + (2)(-2) + (-3)(3) = -1 + -4 + -9 = -14$$

Nothing wrong with having variables as elements of the vectors:

$$(1, 2, 3)^T \cdot (x, y, z)^T = x + 2y + 3z$$

QUESTION 12:

You must be itching to try this yourself (or is that your allergy to math acting up again?)

$$(4, 0, -3)^T \cdot (0, -2, 0)^T = ?$$

Click Here after you have answered the question

$(4, 0, -3)^T \cdot (0, -2, 0)^T = ?$

A good answer might be:

$(4, 0, -3)^T \cdot (0, -2, 0)^T = 4*0 + 0*(-2) + (-3)*0 = 0+0+0 = 0$

Dot Product with Zero Vector

Notice that the length of each vector that went into the dot product of the question was greater than zero, but that the dot product was zero. Here is another example:

$$(0,0,0)^T \cdot (-2.3, 89.22, 0)^T = 0(-2.3) + 0(89.22) + 0(0) = 0$$

This is not a surprise (I hope). We saw the same thing with geometrical vectors.

$$\mathbf{0} \cdot \mathbf{a} = 0$$

This *looks* obvious. The first **0** is the *zero column matrix*; the last 0 is the *real number* zero. Also,

$$\mathbf{0} \cdot \mathbf{0} = 0$$

In each of these equations the zero column matrix means a column matrix of the same dimension as the other column matrix, with each element the real number zero.

QUESTION 13:

More practice:

$$(-2, 5, -6)^T \cdot (1, 2, 3)^T = ?$$

Click Here after you have answered the question

A good answer might be:

$(-2, 5, -6)^T \cdot (1, 2, 3)^T = (-2)*1 + 5*2 + (-6)*3 = -2 + 10 -18 = -10$

Any Real is Possible

The answer is a negative real. This is fine. Dot products produce all kinds of reals: negative, zero, positive. However, something interesting happens when you take the dot product of a vector with itself.

QUESTION 14:

What is:

$$(1, -3)^T \cdot (1. -3)^T = ?$$

Click Here after you have answered the question

A good answer might be:

$(1, -3)^T \cdot (1, -3)^T = (1)(1) + (-3)(-3) = 1 + 9 = 10$

Alway Positive

As with vectors, the dot product of a column matrix with itself will always be positive.

$$(x, y, z)^T \cdot (x, y, z)^T = x^2 + y^2 + z^2 = \text{something zero or positive}$$

No matter what x, y, and z are, their square is going to be zero or greater. So the sum is going to be zero or greater.

The only time the square of a real number is zero is when the real number is zero; so the only time the dot product of a column matrix with itself is zero is when it is the zero column matrix.

QUESTION 15:

g is a 3 dimensional vector. Solve the following equation for the elements of **g**:

$$\mathbf{g} \cdot \mathbf{g} = 0$$

Click Here after you have answered the question

A good answer might be:

By the previous discussion, every element of **g** must be zero. So:

$$\mathbf{g} = \mathbf{0} = (0, 0, 0)^T$$

Distributive with repect to Vector Addition

The dot product is distributive with respect to vector addition:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

This looks reasonable, but be careful that you understand what it says. In particular, the "+" on either side of "=" mean different things.

QUESTION 16:

1. What is the meaning of the "+" on the LEFT side of the equation?
2. What is the meaning of the "+" on the RIGHT side of the equation?

Click Here after you have answered the question

a · (b + c) = a · b + a · c

A good answer might be:

1. What is the meaning of the "+" on the LEFT side of the equation: Vector Addition.
2. What is the meaning of the "+" on the RIGHT side of the equation: Real Number Addition.

Demonstration of Distributive Property

It is fairly easy to demonstrate that the distributive property is true. Let's do the demonstration with 3D column matrices, although it will work with any dimension. Both sides of the equation represent a real number. So the job is to check that both sides represent the *same* real number.

1. Show that: **a · (b + c) = a · b + a · c**
2. Let **a** = $(f, g, h)^T$
3. Let **b** = $(r, s, t)^T$
4. Let **c** = $(x, y, z)^T$
5. **a · (b + c)** = **a** · $(r+x, s+y, t+z)^T$ = $f(r+x) + g(s+y) + h(t+z)$ = $fr + fx + gs + gy + ht + hz$
6. **a · b + a · c** = $(fr + gs + ht) + (fx + gy + hz)$ = $fr + fx + gs + gy + ht + hz$
7. Both sides of the equation represent the same real number.

The same demonstration could be done with vectors of any dimension, so the distributive property is true for any dimension.

QUESTION 17:

What is: $(-2, 1, 2)^T \cdot ((3, -1, 4)^T + (-2, 1, -2)^T)$

Click Here after you have answered the question

A good answer might be:

It is probably easiest NOT to re-arrange the expression, but to do the vector addition first:

$$(-2, 1, 2)^T \cdot ((3, -1, 4)^T + (-2, 1, -2)^T) = (-2, 1, 2)^T \cdot (1, 0, 2)^T$$

$$= -2*1 + 1*0 + 2*2 = -2 + 4 = 2$$

Practice with Distribution

Other times it is easiest to re-arrange the expression.

QUESTION 18:

What is: $(-2, 0, 2)^T \cdot ((1, -1, 1)^T + (3, 1, 4)^T)$

Click Here after you have answered the question

A good answer might be:

Here it is probably easiest to do the dot product first:

$$(-2, 0, 2)^T \cdot ((1, -1, 1)^T + (3, 1, 4)^T)$$

$$= (-2, 0, 2)^T \cdot (1, -1, 1)^T + (-2, 0, 2)^T \cdot (3, 1, 4)^T = 0 + 2 = 2$$

End of the Chapter

You may find it useful to distribute your study time over the following topics:

- Various names by which the dot product is known.
- Definition of the dot product for geometrical vectors.
- Commutative property of the vector dot product.
- Cosine of an angle.
- Dot product of orthogonal vectors.
- Dot product and vector length.
- Properties of the vector dot product (list)
- Dot product of 2D column matrices.
- Commutative property of column matrix dot product.
- Dot product of 3D column matrices.
- Dot product of a column matrix with itself.
- Distributive property of column matrix dot product.

Click on a term to see where it was discussed in this chapter. Remember to click on the "Back" arrow of your browser to get back to this page. The next chapter will discuss how the dot product relates to the length of a vector.

Click here to go back to the main menu.

You have reached the End.

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CHAPTER 8 -- Length, Orthogonality, and the Column Matrix Dot Product

This chapter will explore the relationship between the dot product of a column matrix with itself and the length of the vector it represents. It also discusses how the dot product of column matrices that represent orthogonal vectors is zero.

- Length of vector given by column matrix dot product.
- Unit vectors.
- Unit normal.
- Dot product of orthogonal vectors and their column matrices.
- Constructing a vector orthogonal to a given vector.

QUESTION 1:

Let $\mathbf{v} = (3, 4)^T$

1. $\mathbf{v} \cdot \mathbf{v} = ?$
2. The length of $\mathbf{v} = ?$

Click Here after you have answered the question

A good answer might be:

Let **v** = $(3, 4)^T$

1. **v** · **v** = ?
   - $(3, 4)^T \cdot (3, 4)^T = 3^2 + 4^2 = 9 + 16 = 25 = 5^2$
2. The length of **v** = ?
   - The length of the vector is **5**.

Hmm... there might be a connection here....

Dot Product Formula for Length

As you have seen in the previous chapter:

$$(x, y, z)^T \cdot (x, y, z)^T = x^2 + y^2 + z^2$$

Another way of writing this is:

$$\mathbf{v} \cdot \mathbf{v} = | \mathbf{v} |^2$$

The dot product of a column matrix with itself is a scalar, the square of the length of the vector it represents.

WARNING! When your graphics text starts using *homogeneous coordinates* this calculation will need to be modified somewhat. Remember, length is a property of the geometric vector, not an inherent property of the column matrix that might be used to represent it.

QUESTION 2:

What is the length of the vector represented by $(2, 1, -1)^T$?

Click Here after you have answered the question

A good answer might be:

The dot product is 4+1+1 = 6; so the length is $\sqrt{6}$.

Practice

Here are some easily confirmed facts:

1. $(k\mathbf{v}) \cdot (k\mathbf{v}) = k^2\mathbf{v}^2$ ie. $|\,k\mathbf{A}| = k|\,\mathbf{v}\,|$
2. $|\,\mathbf{0}\,| = 0$
3. $\mathbf{0} \cdot \mathbf{v} = 0$
4. $|\,-\mathbf{v}\,| = |\,\mathbf{v}\,|$

Here are those facts in English:

1. If you scale a vector by a scalar k, you multiply its length by k.
2. The length of the zero vector is zero.
3. The dot product of any vector with the zero vector is zero.
4. A vector's length is the same if it points in the opposite direction.

QUESTION 3:

What is the length of the vector $(30, 40)^T$?

Click Here after you have answered the question

A good answer might be:

$(30, 40)^T = 10(3, 4)^T$, so the length is $10*5 = 50$

Or, if you want to do it the hard way:
$|(30,40)|^2 = 900 + 1600 = 2500 = 25e2$. The length $= \sqrt{25e2} = 5e1 = 50$.

Unit Vectors

Recall that a **unit vector** is a vector of length one. Creating a unit vector in the same direction as a given vector is called **normalizing** the vector. Here is the formula for normalizing a vector (copied from a previous chapter):

$$\mathbf{v_u} = \mathbf{v} / |\mathbf{v}|$$

The formula works when column matrices are used to represent the vector. The length **| v |** is frequently calculated by using the column matrix dot product.

Potential Confusion: The word "normal" has several meanings. Normalizing a vector means making a unit vector in the same direction as the original. A vector that is perpendicular to a particular surface is sometimes called a "normal vector" but is not necessarily a unit vector. Be careful never to say "normal vector" when you mean "unit vector."

QUESTION 4:

Normalize the vector represented by $(1.2, -4.2, 3.5)^T$.

Click Here after you have answered the question

A good answer might be:

The dot product is:

$$(1.2, -4.2, 3.5)^T \cdot (1.2, -4.2, 3.5)^T = 1.2^2 + (-4.2)^2 + 3.5^2 = 31.33.$$

The unit vector is represented by:

$$(1.2, -4.2, 3.5)^T / \sqrt{31.33}.$$

Keep the length Explicit

Usually you should leave an answer in the above form (i.e., don't divide each element by the length.) Often a later calculation will cancel the length if you leave it explicit.

QUESTION 5:

Answer the following questions. (You might want to think of 2D vectors when you answer, although the answers will be correct for 3D vectors as well).

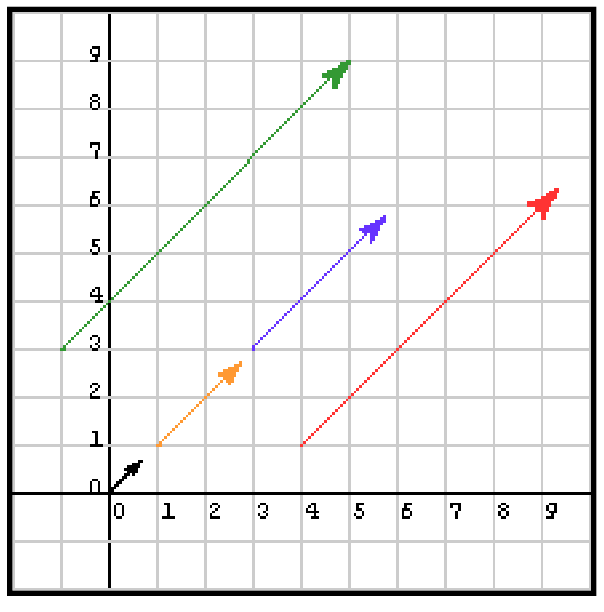
1. How many vectors can there be in a given direction?
2. How many UNIT vectors can there be in a given direction?

Click Here after you have answered the question

A good answer might be:

1. How many vectors can there be in a given direction? Infinitly Many.
2. How many UNIT vectors can there be in a given direction? One.

Unit Vectors as Directions



There can be infinitely many vectors in a given direction since the elements of a vector are real numbers. So, for example in two dimensions, there are an infinite number of vectors that look like (s, 0), all of them pointing along the x-axis.

The unit vector for a given vector is in the same direction as that vector. There is only one unit vector in a given direction since a vector has only one length, so there is only one value for the expression **v**/| **v** |. Continuing the two dimensional example, there is only one unit vector in the positive x direction, represented by $(+1, 0)^T$.

The same is true for other orientations and other number of dimensions. The picture shows vectors of various lengths but same orientation. Only one of them is a unit vector.

Recall the formula:

angle = arc tan(y/x)

This formula is not very useful in three dimensions, where there are three axes and it is not enough to determine the angle between a vector and just one of them.

Often the *unit vector corresponding to a given vector* is used to express the given vector's direction. There is only one corresponding unit vector, so this description of direction is unique. This works for all dimensions.

QUESTION 6:

What is the direction of **v** = $(-3, 2, 4)^T$?

Click Here after you have answered the question

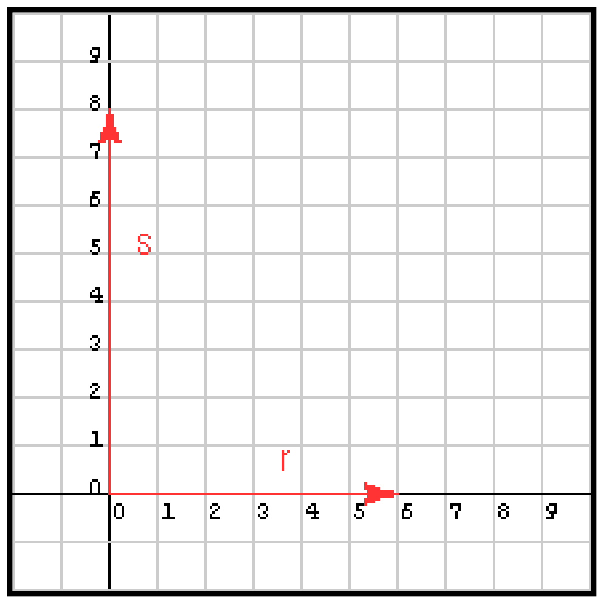
What is the direction of **v** = $(-3, 2, 4)^T$?

A good answer might be:

Calculate the unit vector:

1. $| \mathbf{v} |^2 = 9 + 4 + 16 = 29$
2. $\mathbf{v_u} = \mathbf{v} / \sqrt{29}$

Right Angles



It is interesting when the directions of two vectors are 90° apart. For example, zero degrees and 90 degrees.

QUESTION 7:

What is the dot product of **r** = $(6, 0)^T$ with **s** = $(0, 8)^T$?

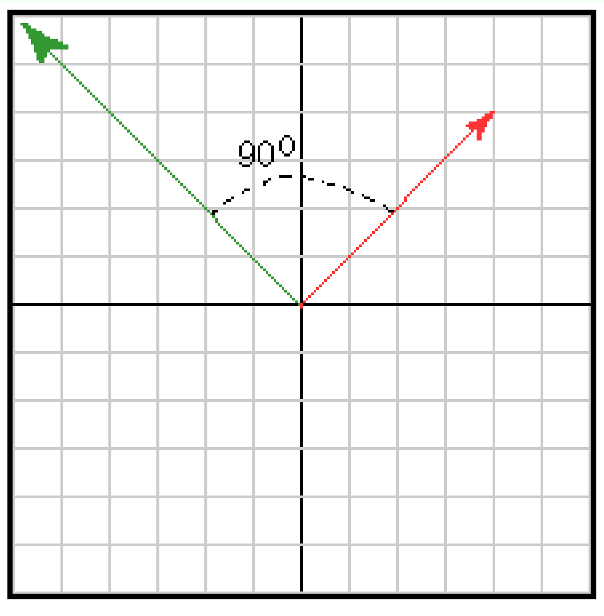
Click Here after you have answered the question

What is the dot product of **r** = $(6, 0)^T$ with **s** = $(0, 8)^T$?

A good answer might be:

$(6, 0)^T \cdot (0, 8)^T = 6*0 + 0*8 = 0$

Same Vectors, Rotated



The question was a particularly easy case. Here are the same two vectors rotated 45°:

- **r'** = $(6\cos(45), 6\sin(45))^T$
- **s'** = $(-8\sin(45), 8\cos(45))^T$

Don't worry too much about how I got the rotatated vectors. This topic will come up soon enough.

QUESTION 8:

Form the dot product of the two vectors:

- **r'** = $(2\cos(45), 2\sin(45))$
- **s'** = $(-3\sin(45), 3\cos(45))$

(Yes, I know, this looks an awful lot like trigonometry. But it isn't. Just do it.)

Click Here after you have answered the question

Form the dot product of vectors represented by:

- **r'** = (2cos(45), 2sin(45))
- **s'** = (-3sin(45), 3cos(45))

A good answer might be:

Plugging away:

r' · **s'** = 2cos(45) * (-3)sin(45) + 2sin(45) * 3cos(45)
= -2cos(45) * (3)sin(45) + 2sin(45) * 3cos(45) = -6cos(45) * sin(45) + 6cos(45) * sin(45)
= 0

Dot Product of Orthogonal Vectors

It seems reasonable that the dot product of two vectors is the same after they both have been rotated by the same amount.

> The dot product of two orthogonal vectors is zero. The dot product of the two column matrices that represent them is zero.

Only the relative orientation matters. If the vectors are orthogonal, the dot product will be zero. Two vectors do not have to intersect to be orthogonal. (Since vectors have no location, it really makes little sense to talk about two vectors intersecting.)

Of course, this is the same result as before with geometrical vectors.

QUESTION 9:

Are the following orthogonal?

- **q** = $(-5, 3)^T$
- **r** = $(3, 5)^T$

Click Here after you have answered the question

- **q** = $(-5, 3)^T$
- **r** = $(3, 5)^T$

A good answer might be:

Yes, the dot product is zero.

Independent of Length

To form a 2D column matrix that is perpendicular to an other:

Swap the elements, and negate one of them.

This only works in 2D, however. It gives you one of *an infinite number* columns orthogonal to the given one. For example, all the following vectors are orthogonal to $(-5, 3)^T$:

- $(3, 5)^T$
- $(-3, -5)^T$
- $(1.5, 2.5)^T$
- $(6, 10)^T$
- .... and so on

The reason this is so is: If $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$. So $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = 0$, for any real number k. So there are an infinite number of vectors ($k\mathbf{u}$) orthogonal to $\mathbf{v}$.

Often one wishes to find a *unit normal* to a given vector. A **unit normal** to a given vector is a vector that:

1. Is orthogonal (normal) to the given vector.
2. Has a length of one.

Remember not to confuse the two ideas *normalizing a vector* (making a unit vector in the same direction as the vector), and *computing a unit normal* (making a unit vector in an orthogonal direction to a vector.)

QUESTION 10:

Calculate a unit normal to $(-5, 3)^T$.

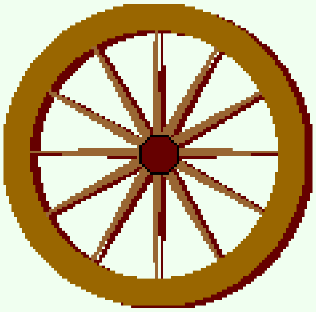
Click Here after you have answered the question

Calculate a unit normal to $(-5, 3)^T$.

A good answer might be:

$(3, 5)/(\sqrt{34})$. Or you might pick the other one (there are two): $(-3, -5)/(\sqrt{34})$.

Infinitely Many Normals



Vectors in three dimensions, also, have unit normals. But a unit normal vector to a given vector is not unique. Think of the axle of a wagon wheel. All the spokes are orthogonal to the axle. Any line in the plane of the spokes is orthogonal to the axle; and there are an infinite number of them.

QUESTION 11:

Find several normals to the vector represented by $(1, 2, 2)^T$.

Click Here after you have answered the question

Find several normals to the vector represented by $(1, 2, 2)^T$.

A good answer might be:

- (0, 0, 0)
- (-2, 1, 0)
- (0, -1, 1)
- (-4, 1, 1)
- (0, 1, -1)
- ... and infinitely many more

These are all normal to the given vector because their dot product with it is zero. These are not unit normals. But even if you were to insist on unit normals, there would still be an infinite number.

Zero Vector as Orthogonal

Notice the first answer (above). The dot product of the zero vector with the given vector is zero, so the zero vector must be orthogonal to the given vector. This is OK. Math books often use the fact that the zero vector is orthogonal to every vector (of the same type). We probably won't need this; but future math courses might. Of course, the length of the zero vector is zero, so it is not a unit normal.

From analytic geometry you may recall the formula for the slope of a line in two dimensions not parallel to the Y axis:

m = (change in y) / (change in x)

This is the same formula as for the tangent of the angle with the x axis. This formula is related to the reason that the dot product of orthogonal 2D vectors is zero.

QUESTION 12:

- What is the slope of $(2, 5)^T$? (Call it m_1)
- What is the slope of $(-5, 2)^T$? (Call it m_2)
- What is the dot product?
- What is m_1 times m_2 ?

Click Here after you have answered the question

A good answer might be:

- What is the slope of $(2, 5)^T$? $m_1 = 5/2$
- What is the slope of $(-5, 2)^T$? $m_2 = -2/5$
- What is the dot product? $(2)(-5) + (5)(2) = 0$
- What is m_1 times m_2? $5/2 * -2/5 = -1$

Slopes of Perpendicular Lines

You may recall this from past math classes:

> If two lines are perpendicular, then the product of their slopes is -1.

This is another way to look at what goes on when you make a normal vector to a given 2D vector by swapping elements and negating one:

- If $\mathbf{v} = (x, y)^T$
- Then $\mathbf{v'} = (-y, x)^T$ is orthogonal,
- because $(x, y)^T \cdot (-y, x)$ is $-xy + yx = 0$.
- The slope of $(x, y)^T$ is y/x.
- The slope of $(-y, x)^T$ is $-x/y$.
- The product of the slopes is $y/x * -x/y = -(xy)/(xy) = -1$

QUESTION 13:

Are $(-1.5, 6)^T$ and $(2, 2)^T$ orthogonal?

Click Here after you have answered the question

Are (-1.5, 6)^T and (2, 2)^T orthogonal?

A good answer might be:

No: the dot product is -3 + 6 = 3.

End of the Chapter

Since the dot product is not zero, the two vectors are not orthogonal. You don't even have to think about lengths. (Remember that length does not matter in this test for orthogonality.) The next chapter will discuss what a non-zero dot product of two vectors gives you.

You have reached the end of the chapter. Normally at this point you would review the following terms:

- Length of a vector as a dot product.
- Some facts about the dot product and vector length.
- Unit vector (review.)
- Easy way to get confused.
- Unit vector used to express direction.
- Dot product of orthogonal vectors.
- Way to construct a 2D vector orthogonal to another one.
- Unit Normal.
- Unit normal in 3D.
- Definition of the slope of a 2D line (review.)
- Relationship between slopes of perpendicular 2D lines.

Click on a term to see where it was discussed in this chapter. Remember to click on the "Back" arrow of your browser to get back to this page.

Click here to go back to the main menu.

You have reached the End.

created 08/26/97; revised 08/13/00

CHAPTER 9 --- The Angle Between Two Vectors

The previous chapter discussed some uses of the dot product---computing the length of a vector, and determining if two vectors are perpendicular:

- **Length:** dot product of vector with itself = length^2
- **Orthogonality:** dot product of two perpendicular vectors = 0

In the first case, the angle between the vectors and itself is 0°. In the second case, the angle between the two vectors is 90°.

QUESTION 1:

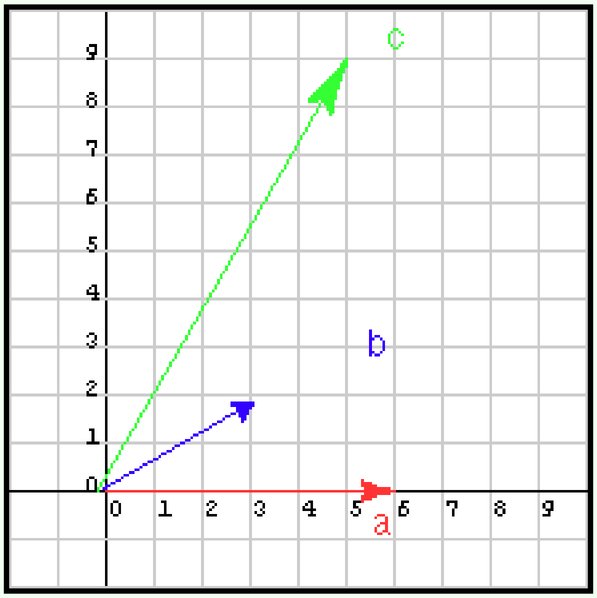
Does the angle between two vectors have something to do with the dot product?

Click Here after you have answered the question

A good answer might be:

Yes---maybe the dot product is large when the angle is close to zero and small when the angle is close to perpendicular.

Example Dot Products



The answer is correct, but a little vague. Let us tighten it up. For now, think about 2D vectors. Consider vector **a** in the diagram and its dot product with two other vectors, **b** oriented at 30°, and **c** oriented at 60°.

QUESTION 2:

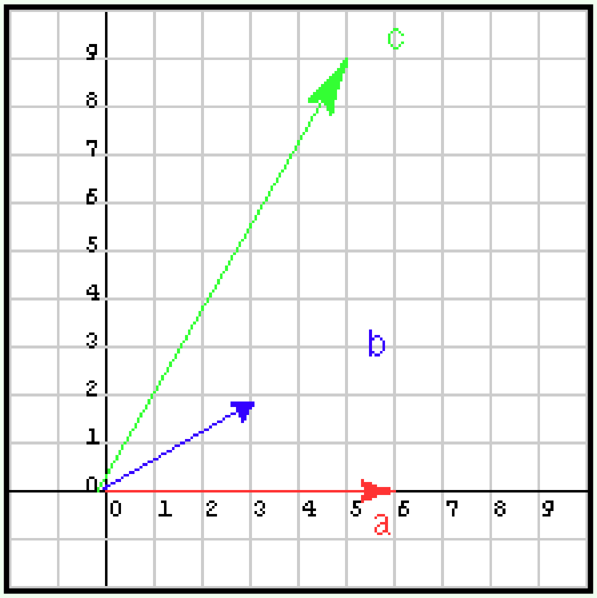
Which do you think is larger: **a** · **b** , or **a** · **c** ?

Click Here after you have answered the question

A good answer might be:

c is much longer vector than **b**; it is likely that **a** · **c** is the larger.

Length Affects Dot Product



The answer is not only likely, but correct.

- **a** is (6, 0)
- **b** is (3, 1.9)
- **c** is (5, 9)

The dot products are:

- **a** · **b** is 15.0
- **a** · **c** is 30.00

So there are two properties tangled up in the dot product: the angle between the input vectors, and the length of the input vectors. We would like to get at just the angle alone by somehow removing the influence of length.

QUESTION 3:

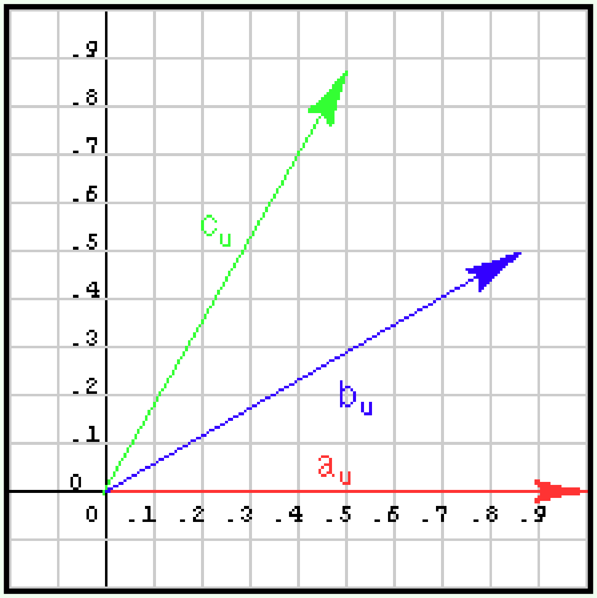
What is the normal method for removing the influence of length?

Click Here after you have answered the question

A good answer might be:

Normalize the vectors (create unit vectors in the same directions).

Normalize Vectors



The previous diagram now shows each vector after it has been *normalized*. (Remember, this means creating a unit vector in the same direction as the original vector).

QUESTION 4:

Now which dot product is larger?

- **a** · **b** , or
- **a** · **c** ?

Hint: remember that u · v = |u| |v| cos θ. But here both vectors have length 1.0.

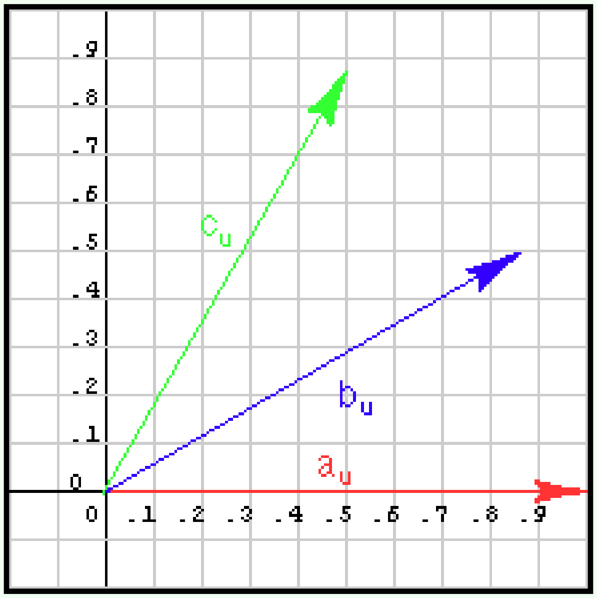
Click Here after you have answered the question

A good answer might be:

Since u · v = |u| |v| cos θ and all vectors in this problem have length 1.0, u · v = cos θ. The dot product of $\mathbf{a_u} \cdot \mathbf{b_u}$ is the cosine of the angle between $\mathbf{a_u}$ and $\mathbf{b_u}$, which can be read off the diagram as 0.866.

The dot product of $\mathbf{a_u} \cdot \mathbf{c_u}$ is the cosine of the angle between $\mathbf{a_u}$ and $\mathbf{c_u}$, which can be read off the diagram as 0.500.

Filtering out the effect of Length



$\mathbf{b_u}$ is closer in orientation to $\mathbf{a_u}$; so $\mathbf{a_u} \cdot \mathbf{b_u}$ is the larger.

- $\mathbf{a_u}$ is (1, 0) --- a unit vector at zero degrees
- $\mathbf{b_u}$ is (0.866, 0.5) --- a unit vector at 30 degrees
- $\mathbf{c_u}$ is (0.5, 0.866) --- a unit vector at 60 degrees

Remember: cos 30° = 0.866, sin 30° = 0.5, cos 60° = 0.5, and sin 60° = 0.866.
The dot products are:

- $\mathbf{a_u} \cdot \mathbf{b_u}$ is 0.866
- $\mathbf{a_u} \cdot \mathbf{c_u}$ is 0.500

So by using vectors of length one, the effect of length is removed and the dot product is larger when a small angle separates the vectors.

QUESTION 5:

What do you suppose happens when the vectors are in opposite directions, such as $(1, 0)^T$ and $(-1, 0)^T$?

Click Here after you have answered the question

What do you suppose happens when the vectors are in opposite directions, such as $(1, 0)^T$ and $(-1, 0)^T$?

A good answer might be:

The magnitude of the dot product is negative.

Range of the Dot Product of Two Unit Vectors

Here is a sampling of $\mathbf{b_u}$ and the dot product with $\mathbf{a_u} = (1.0, 0)^T$ for various angles.

Angle	b	Result		Angle	b	Result
000°	$(1.000, 0.000)^T$	1.000		195°	$(-0.966, -0.259)^T$	-0.966
015°	$(0.966, 0.259)^T$	0.966		105°	$(-0.259, 0.966)^T$	-0.259
030°	$(0.866, 0.500)^T$	0.866		210°	$(-0.866, -0.500)^T$	-0.866
045°	$(0.707, 0.707)^T$	0.707		225°	$(-0.707, -0.707)^T$	-0.707
060°	$(0.500, 0.866)^T$	0.500		240°	$(-0.500, -0.866)^T$	-0.500
075°	$(0.259, 0.966)^T$	0.259		255°	$(-0.259, -0.966)^T$	-0.259
090°	$(0.000, 1.000)^T$	0.000		270°	$(0.000, -1.000)^T$	0.000
120°	$(-0.500, 0.866)^T$	-0.500		285°	$(0.259, -0.966)^T$	0.259
135°	$(-0.707, 0.707)^T$	-0.707		300°	$(0.500, -0.866)^T$	0.500
150°	$(-0.866, 0.500)^T$	-0.866		315°	$(0.707, -0.707)^T$	0.707
175°	$(-0.966, 0.259)^T$	-0.966		330°	$(0.866, -0.500)^T$	0.866
180°	$(-1.000, 0.000)^T$	-1.000		345°	$(0.966, -0.259)^T$	0.966

The $\mathbf{b_u}$ in each case is the unit vector represented by $(\cos \theta, \sin \theta)^T$.

QUESTION 6:

What do you imagine is the range of values for the dot product of two unit vectors, $\mathbf{a_u} \cdot \mathbf{b_u}$?

Click Here after you have answered the question

A good answer might be:

Looks like the range is -1.0 ... 0.0 ... 1.0

Formula for the Angle between Two Vectors

The range is minus one to plus one, because each dot product in the previous page is:

$$(1,\ 0)^T \cdot (\cos\theta, \sin\theta)^T = \cos\theta$$

This is true when $\mathbf{a_u}$ is a unit vector pointing in any direction

The angle between two unit vectors:

$$\mathbf{a_u} \cdot \mathbf{b_u} = \cos\theta$$

where θ is the angle between the two vectors. This formula automatically includes the fact that the dot product of perpendicular vectors is zero (because cos 90° is zero).

QUESTION 7:

What is the cosine of the angle between the two unit vectors $(0.7071, 0.7071)^T$ and $(0.5, 0.866)^T$? (You may wish to use the desktop's calculator for this).

Click Here after you have answered the question

What is the cosine of the angle between the two unit vectors $(0.7071, 0.7071)^T$ and $(0.5, 0.866)^T$?

A good answer might be:

$\cos\theta = (0.7071, 0.7071)^T \cdot (0.5, 0.866)^T = 0.7071*0.5 + 0.7071*0.866 = 0.9659$

Practice

The first vector, $(0.7071, 0.7071)^T$ is a unit vector at 45°. The second vector, $(0.5, 0.866)^T$ is a unit vector at 60°. So the dot product should be the cosine of 15°, which it is.

Another way to see this is with symbols.

1. The first vector is $(\cos 45, \sin 45)^T$
2. The second vector is $(\cos 60, \sin 60)^T$
3. The dot product is cos 45 cos 60 + sin 45 sin 60
4. $= \cos 45 \cos(45 + 15) + \sin 45 \sin(45+15)$
5. $= \cos 45 (\cos 45 \cos 15 - \sin 45 \sin 15) +$
   $= \sin 45 (\cos 45 \sin 15 + \cos 15 \sin 45)$
6. $= \cos 45 \cos 45 \cos 15 - \cos 45 \sin 45 \sin 15 +$
   $= \sin 45 \cos 45 \sin 15 + \sin 45 \cos 15 \sin 45$
7. $= \cos^2 45 \cos 15 + \sin^2 45 \cos 15$
8. $= (\cos^2 45 + \sin^2 45)\cos 15$
9. $= \cos 15$

Step 5 makes use of the formulae:

$$\cos(x+y) = \cos(x) \cos(y) - \sin(x) \sin(y)$$
$$\sin(x+y) = \sin(x) \cos(y) + \cos(x) \sin(y)$$

Step 9 makes use of the formula:

$$\sin^2 x + \cos^2 x = 1$$

QUESTION 8:

What is the cosine of the angle between the two unit vectors $\mathbf{q_u} = (0.0, 1)^T$ and $\mathbf{r_u} = (0.5, 0.866)^T$?

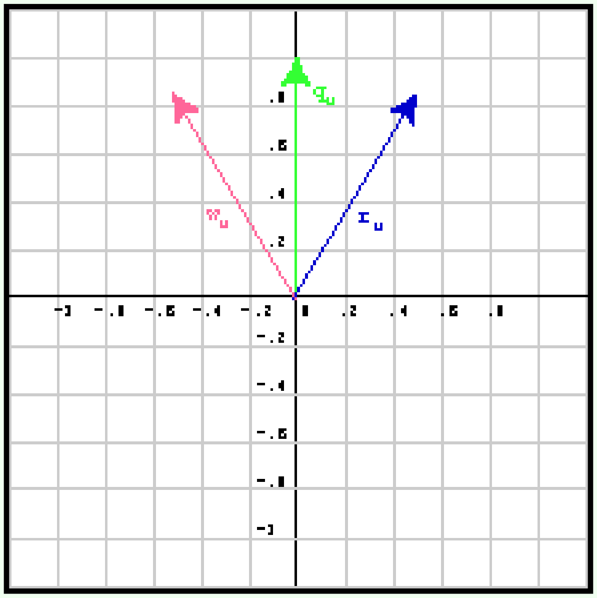
Click Here after you have answered the question

$\mathbf{q_u} = (0.0, 1)^T$ and $\mathbf{r_u} = (0.5, 0.866)^T$?

A good answer might be:

$\mathbf{q_u} \cdot \mathbf{r_u} = \cos\theta = 0.866.$ So $\theta = \text{arc}\cos 0.866 = 30°.$

More Practice



The picture shows $\mathbf{q_u}$ and $\mathbf{r_u}$ and the angle between them.

Since $\mathbf{q_u} \cdot \mathbf{r_u} = \mathbf{r_u} \cdot \mathbf{q_u}$, the angle is the angle < 180° between the two vectors. It does not matter which vector is first in the dot product.

Sometimes (especially when vectors point into differing quadrants) itreally helps to draw a picture to check that the answer you compute makes sense.

QUESTION 9:

What is the cosine of the angle between the two unit vectors $\mathbf{q_u} = (0.0, 1)^T$ and $\mathbf{w_u} = (-0.5, 0.866)^T$?

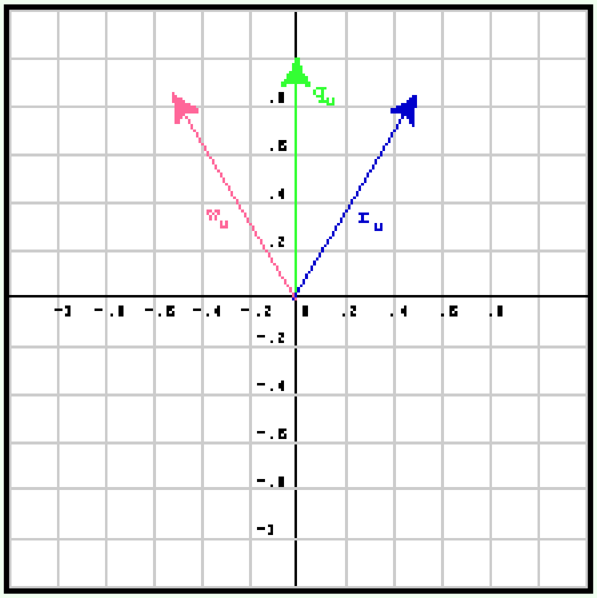
[Click Here after you have answered the question]

What is the cosine of the angle between the two unit vectors $\mathbf{q_u} = (0.0, 1)^T$ and $\mathbf{w_u} = (-0.5, 0.866)^T$?

A good answer might be:

Again, cos x = 0.866. So it must be that x = arc cos 0.866 = 30°.

Which Side?



Both vectors $\mathbf{r_u}$ and $\mathbf{w_u}$ gave the answer cos 30, when the dot prodcut was done with $\mathbf{q_u}$, although they lie on either side of $\mathbf{q_u}$.

In two dimensions, there are two unit vectors that are 30° away from a given vector. Both of them will give you the same dot product with the given vector. Be careful to draw a picture in ambiguous situations.

QUESTION 10:

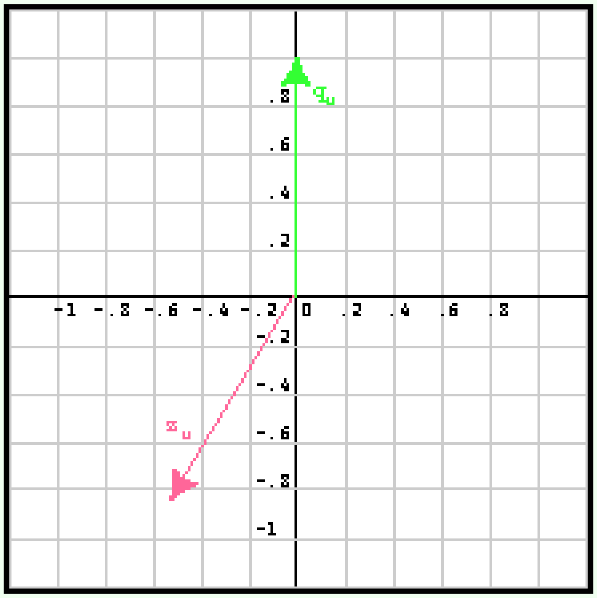
What is the cosine of the angle between the two unit vectors $\mathbf{q_u} = (0.0, 1)^T$ and $\mathbf{z_u} = (-0.5, -0.866)^T$?

Click Here after you have answered the question

A good answer might be:

Now, $\cos\theta = -0.866$. So $\theta = \arccos -0.866 = 150°$.

Angles Greater than 90



$\mathbf{q_u} = (0.0, 1)^T$ and $\mathbf{z_u} = (-0.5, -0.866)^T$

The angle between the two vectors is greater than 90°, so the cosine is negative.

QUESTION 11:

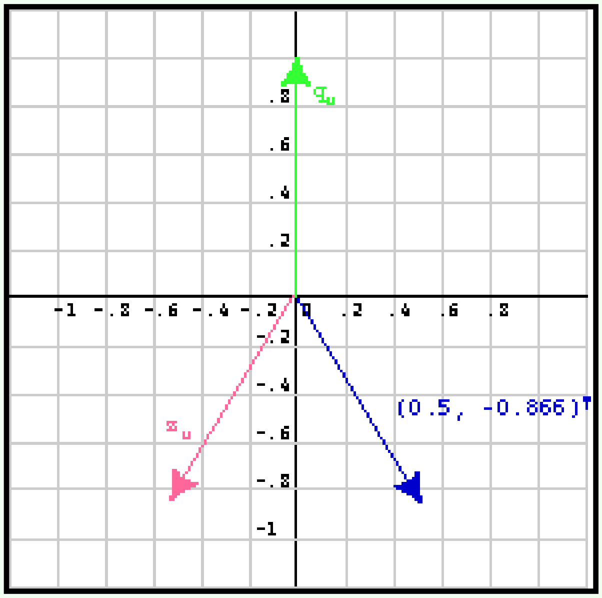
In the diagram above, mentally draw *the other* vector that has a dot product of -0.866 with vector $\mathbf{q_u}$.

Click Here after you have answered the question

A good answer might be:

The picture is below.

Same Angle; Same Dot Product



The two vectors that have a dot product of -0.866 with $\mathbf{q_u} = (0.0, 1)^T$ are:

- $\mathbf{z_u} = (-0.5, -0.866)^T$
- $\mathbf{v_u} = (0.5, -0.866)^T$

In the diagram, the two have their tail placed at the origin. (But remember: vectors are not fixed to any particular location and can be drawn anywhere that is convenient). The dot product gives you the angle between the orientation of two vectors.

QUESTION 12:

What is the angle between these two vectors: $\mathbf{d_u} = 0.7071(1, 1)^T$ and $\mathbf{e_u} = -0.7071(1, 1)^T$

Sketch the vectors to see the situation.

Click Here after you have answered the question

What is the angle between these two vectors: $\mathbf{d_u} = 0.7071(1, 1)^T$ and $\mathbf{e_u} = -0.7071(1, 1)^T$

A good answer might be:

$$0.7071(1, 1)^T \cdot -0.7071(1, 1)^T = 0.7071(-0.7071)(1+1)$$
$$= -0.7071^2 * 2 = -0.5*2 = -1$$

so the angle is: arc cos(-1) = 180°. In 2D, there is only one vector 180° away from a given vector.

More Practice

Of course, the real world rarely gives you easy angles of 30°, 45°, 60°, and so on. Here is a more realistic problem: Find the angle between:

- $\mathbf{f_u} = (0.6, 0.8)^T$
- $\mathbf{g_u} = (0.8, 0.6)^T$

You may wish to confirm that these are indeed unit vectors. Now, sketch the vectors. Then, calculate their dot product. Use the arc cos function of the desktop calculator (or a real calculator) to find the angle. Calculate the answer in degrees, not radians.

QUESTION 13:

What is the angle?

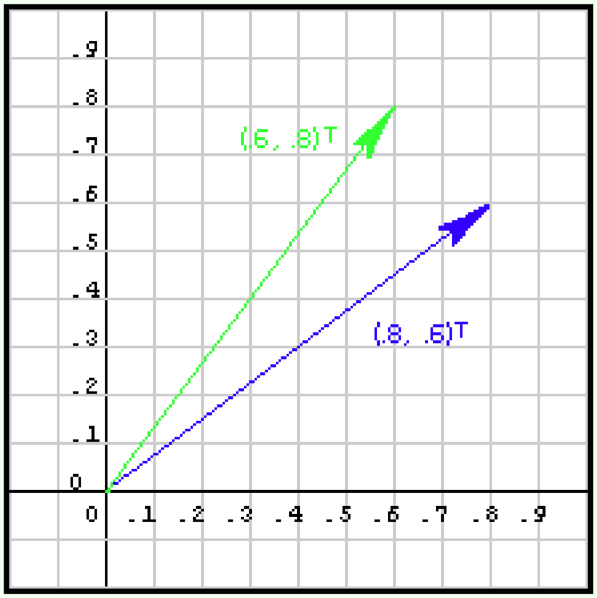
Click Here after you have answered the question

A good answer might be:

The picture is below.

- The dot product is (0.6, 0.8) · (0.8, 0.6) = 0.6*0.8 + 0.8*0.6 = 0.48 + 0.48 = 0.96
- The angle is arc cos(0.96) = 16.26°

Non-unit Vectors



Consider the following problem: Find the angle between the two vectors:

- **j** = $(3, 4)^T$
- **k** = $(0, 2)^T$

These are not *unit vectors*, so you can not find the angle between them without further work. As always, draw a sketch first.

QUESTION 14:

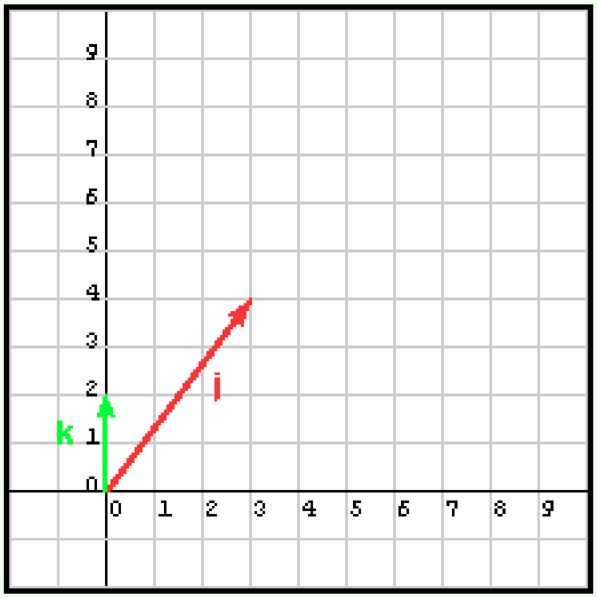
What do you suggest doing, so that the angle between the two vectors can be found?

Click Here after you have answered the question

A good answer might be:

Normalize each vector. This does not change the orientations. Now there are two unit vectors and the cosine of the angle between them can be found with the dot product.

Angle between Non-unit Vectors



Find the angle between the two vectors: $\mathbf{j} = (3, 4)^T$, $\mathbf{k} = (0, 2)^T$

First, find the length of each vector: $|\,\mathbf{j}\,| = 5.0$, $|\,\mathbf{k}\,| = 2.0$.

Normalize each vector. Often in this step it is wise to keep the fractions: $\mathbf{j_u} = (3, 4)^T /5.0$, $\mathbf{k_u} = (0, 2)^T /2.0$.

Compute the dot product: $(3, 4)^T /5.0 \cdot (0, 2)^T / 2.0 = (1/10)(3, 4)^T \cdot (0, 2)^T = (0.1)(8) = 0.8$.

Finally use the arc cos function of your calculator: arc cos 0.8 = 36.87°

QUESTION 15:

Calculate the angle between: $\mathbf{a} = (10, 5)^T$ and $\mathbf{b} = (8, 12)^T$

Click Here after you have answered the question

Calculate the angle between: **a** = $(10, 5)^T$ and **b** = $(8, 12)^T$

A good answer might be:

The lengths are: | **a** | = $\sqrt{125}$, | **b** | = $\sqrt{208}$. The dot product is:

$$(10, 5)^T/\sqrt{125} \cdot (8, 12)^T/\sqrt{208} = (10*8 + 5*12) / (\sqrt{125}\sqrt{208})$$
$$= 140 / (\sqrt{125}\sqrt{208}) = 140/\sqrt{(125*208)}$$
$$= 140/\sqrt{26000} = 0.86824$$

so the angle is: arc cos (0.86824) = 29.74°

End of the Chapter

This chapter had quite a bit of numerical computation in it. Be careful that you did not loose sight of the important points. You may wish to go through this list to see what the important points were:

- Two properties that are tangled together in the dot product.
- Formula for the angle between two unit vectors.
- How many vectors have the same angular separation from a given vector?
- What a negative cosine means for angular separation.
- How to find the angle between non-unit vectors.

Click on a term to see where it was discussed in this chapter. Remember to click on the "Back" arrow of your browser to get back to this page.

Click here to go back to the main menu.

You have reached the End.

created 08/26/97; revised 08/14/00

CHAPTER 10 --- Angle between 3D Vectors

We have used the dot product in several calculations:

- **Length:** Dot product of vector with itself = length^2
- **Perpendicularity:** Dot product of perpendicular vectors = 0
- **Angle:** cos θ = dot product of two unit vectors

This chapter adds nothing new, but further elaborates the third application by discussing the angle between vectors in 3D space.

QUESTION 1:

What is the angle between two edges of a cardboard box that meet in a corner?

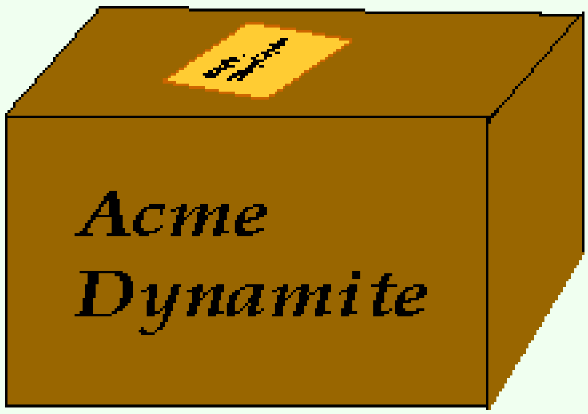
Click Here after you have answered the question

What is the angle between two edges of a cardboard box that meet in a corner?

A good answer might be:

90°

Angles in Three Dimensions



Any two of the three edges of a corner of a cardboard box lie in a plane. The angle between them is 90°.

All the edges of the box intersect at right angles, although in the 2D picture the angles between the 2D lines are not 90°. The 2D picture is a *projection* of the 3D objects onto a plane. The angle between the projection of two vectors will not usually be the same as between the vectors in 3D.

The vertices of the box are points in 3D. The orientation and length of the edges are 3D vectors.

QUESTION 2:

If our viewpoint changes, do the angles between the edges of the box change?

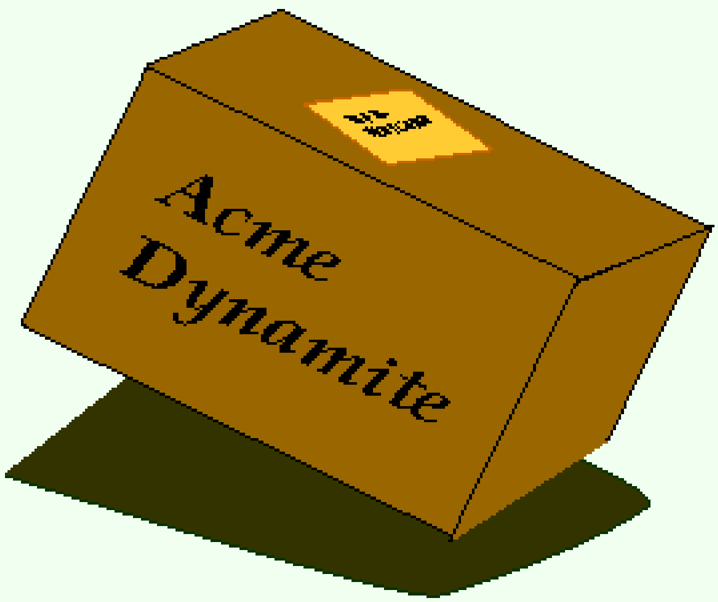
Click Here after you have answered the question

If our viewpoint changes, do the angles between the edges of the box change?

A good answer might be:

No. The orientations and lengths of the edges are 3D vectors. Changing our viewpoint does not change them.

Angles between Coordinate Axes



The angle between two vectors in 3D depends on their relation to each other (and does not depend on our viewpoint, or the coordinate frame we are using). Of course, to give a 3D effect on 2D paper the angle between their 2D counterparts depends on the viewpoint.

In the previous chapter you saw a formula for the angle θ between two unit vectors:

$$\mathbf{a_u} \cdot \mathbf{b_u} = \cos\theta$$

QUESTION 3:

Apply this formula to the two unit vectors parallel to the x axis and y axis (in the coordinate frame we happen to be using): $(1,0,0)^T$ and $(0,1,0)^T$.

Click Here after you have answered the question

Calculate the angle between the x axis and the y axis.

A good answer might be:

$(1,0,0)^T \cdot (0,1,0)^T = 0$. Therefore the x and y axis are orthogonal. (As we expected).

Angle between two Vectors

The previous problem was particularly easy to visualize, and particularly easy to calculate. Let us try something harder: The picture shows two vectors:

Vector **a** is represented by $(4,4,4)^T$
Vector **b** is represented by $(4,0,4)^T$

The angle between the two vectors is not 90°, and so is harder to see than in the previous example. Click on the right and left rotate buttons to get a better idea of the angle.

QUESTION 4:

Take a guess: what angle is between **a** and **b**? 10° ? 35° ? 70° ? 120° ?

Click Here after you have answered the question

What angle is between **a** = $(4,4,4)^T$ and **b** = $(4,0,4)^T$?

A good answer might be:

Visually, it looks somewhat less than midway between perpendicular and horizontal. 35° would be a good guess.

Formula for the angle between two Vectors

One could do better than guess by noticing that in going from the tail to the head of **a** the vertical distance increases by 4 while the horizontal distance increases by $4\sqrt{2}$. Hence the tangent of the angle is $4/(4\sqrt{2}) = 1.0/\sqrt{2} = 0.7071$, so the angle with the horizontal is arctan(0.7071) = 35.26°. Since **b** is in the horizontal plane, the angle between the two vectors must be 35°.

The formula for the angle θ between two unit vectors is:

$$\mathbf{a_u} \cdot \mathbf{b_u} = \cos\theta$$

To use this formula with non-unit vectors: (1) normalize each vector, (2) compute the dot product, (3) take the arc cos to get the angle.

QUESTION 5:

Apply this formula to **a** and **b** of the figure.

Click Here after you have answered the question

Vector **a** is $(4,4,4)^T$ Vector **b** is $(4,0,4)^T$ Calculate: $\mathbf{a_u} \cdot \mathbf{b_u} = \cos\theta$

A good answer might be:

$$|\mathbf{a}| = \sqrt{(16 + 16 + 16)} = 4\sqrt{3}, \quad |\mathbf{b}| = \sqrt{(16 + 16)} = 4\sqrt{2}$$
$$\mathbf{a_u} = (4, 4, 4)^T/(4\sqrt{3}), \quad \mathbf{b_u} = (4, 0, 4)^T/(4\sqrt{2})$$
$$\mathbf{a_u} \cdot \mathbf{b_u} = (16 + 16)/((4\sqrt{3})(4\sqrt{2})) = 2/(\sqrt{3}\sqrt{2}) = \sqrt{2}/\sqrt{3} = \cos\theta$$
$$\cos\theta = 0.81649, \quad \theta = 35.26°$$

A More Difficult Problem

The nasty math in the previous exercise is not the real purpose of all this. The goal is to illustrate the formula $\mathbf{a_u} \cdot \mathbf{b_u} = \cos\theta$, which is important in every part of 3D graphics. It is worth another example.

The figure shows two vectors, represented by:

- $\mathbf{f} = (4,3,2)^T$
- $\mathbf{g} = (-1,4,4)^T$

Rotate the figure to get a better sense of the angle. What you would like to do is to lay a sheet of paper across the two vectors, trace them onto the paper, and then measure the angle with a protractor. But this is hard to do with a computer screen.

QUESTION 6:

Guessing, however, is easy. About what angle separates the two vectors?

Click Here after you have answered the question

About what angle separates the two vectors?

A good answer might be:

After rotating them figure a few times, *I* guessed 50°. But this is only a guess.

Applying the Formula

Lets see how close my guess is:

1. The two vectors are represented by:
   $\mathbf{f} = (4, 3, 2)^T$
   $\mathbf{g} = (-1, 4, 4)^T$
2. The lengths are:
   $|\mathbf{f}|^2 = (4, 3, 2)^T \cdot (4, 3, 2)^T = 16 + 9 + 4 = 29$
   $|\mathbf{g}|^2 = (-1, 4, 4)^T \cdot (-1, 4, 4)^T = 1 + 16 + 16 = 33$
3. The normalized vectors are:
   $\mathbf{f_u} = (4, 3, 2)^T / \sqrt{29}$
   $\mathbf{g_u} = (-1, 4, 4)^T / \sqrt{33}$
4. The dot product is:
   $\mathbf{f_u} \cdot \mathbf{g_u} = (4, 3, 2)^T \cdot (-1, 4, 4)^T / (\sqrt{29}\sqrt{33})$
   $= (-4 + 12 + 8)/(\sqrt{29}\sqrt{33}) = 16 / (\sqrt{29}\sqrt{33}) = 0.51721$
5. The angle is:
   $\cos\theta = 0.51721$
   $\theta = \text{arc cos } 0.51721 = 58.855°$

Not too far off from my guess.

QUESTION 7:

Do two vectors need to be touching at their tails for there to be an angle between them?

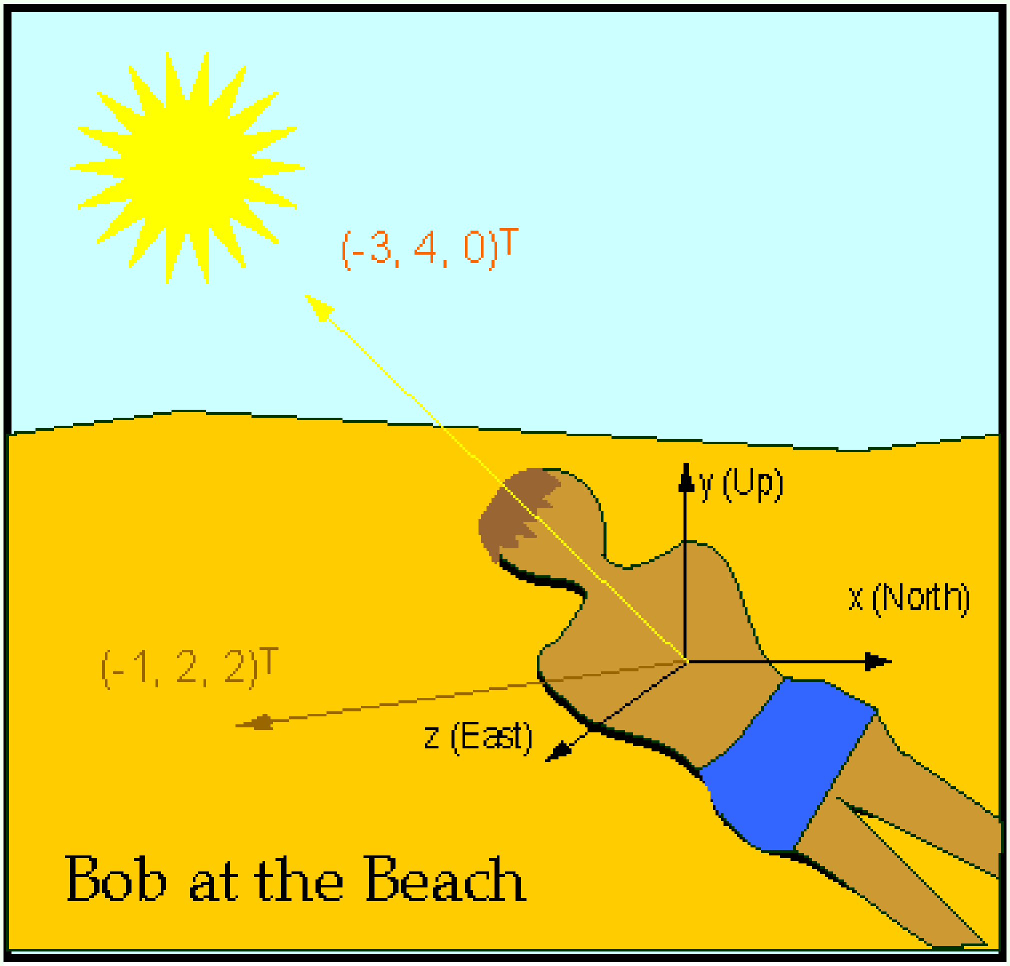
Click Here after you have answered the question

Do two vectors need to be touching at their tails for there to be an angle between them?

A good answer might be:

No---remember vectors don't really have a position. You can take the dot product of any two 3D vectors or the column matrices that represent them.

Angle between Two Directions



It might sound odd to measure the angle between things that have no position. However, many phenomena have an orientation, but no position. The wind might come from the West one day and from the North the next (a difference of 90°). At noon in Summer the light from the Sun arrives at a different angle than at noon in Winter. Shading in 3D graphics uses vectors for the orientation of a surface and for the direction of each light source.

Bob and Bill (two Business students) have skipped class and are at the beach working on their sun tans. A convenient coordinate frame points the x-axis North, the y-axis straight up, and the z-axis East. In this frame, the direction to the Sun is $(-3, 4, 0)^T / 5$. A unit vector perpendicular to Bob's back is $(-1, 2, 2)^T / 3$. A unit vector perpendicular to Bill's back is $(-2, 1, 2)^T / 3$.

QUESTION 8:

Which student is best positioned for rapid tanning?

Click Here after you have answered the question

A good answer might be:

- Direction to sun = (-3, 4, 0)/5
- Perpendicular to Bob's back = (-1, 2, 2)/3
- Perpendicular to Bill's back = (-2, 1, 2)/3

To catch the most rays, your back should be pointed straight at the sun. You want the *smallest* angle between the perpendicular to your back and the direction to the sun.

For Bob: (-1, 2, 2)/3 · (-3, 4, 0)/5 = (3+8+0)/15 = 11/15. arc cos(11/15) = 42.8 degrees
For Bill: (-2, 1, 2)/3 · (-3, 4, 0)/5 = (6+4+0)/15 = 10/15. arc cos(14/15) = 48.2 degrees

Bob will tan faster.

More Practice

The diagram shows two more example vectors:

$\mathbf{q} = (-2, 4, 3)^T$
$\mathbf{p} = (3, 1, -4)^T$

For ease in visualization the tails of the vectors have been placed at the origin. The vectors point into two different octants of 3D space.

Rotate the viewpoint to get a better idea of the two vectors. See if you can find a viewpoint that best shows the angle between the vectors.

QUESTION 9:

What is the angle between the two vectors? (Work it out numerically.)

[Click Here after you have answered the question]

A good answer might be:

120.654 degrees

The Answer Worked Out

In tedious detail:

1. The two vectors are:
   $\mathbf{p} = (-2, 4, 3)^T$
   $\mathbf{q} = (3, 1, -4)^T$
2. The lengths are:
   $| \mathbf{p} |^2 = (-2, 4, 3)^T \cdot (-2, 4, 3)^T = 4 + 16 + 9 = 29$
   $| \mathbf{q} |^2 = (3, 1, -4)^T \cdot (3, 1, -4)^T = 9 + 1 + 16 = 26$
3. The normalized vectors are:
   $\mathbf{p_u} = (-2, 4, 3)^T / \sqrt{29}$
   $\mathbf{q_u} = (3, 1, -4)^T / \sqrt{26}$
4. The dot product is:
   $\mathbf{p_u} \cdot \mathbf{q_u} = (-2, 4, 3)\cdot(3, 1, -4)^T / (\sqrt{29}\sqrt{26})$
   $= (-6 + 4 - 12)/(\sqrt{29}\sqrt{26}) = -14/(\sqrt{29}\sqrt{26}) = -0.50985$
5. The angle is:
   $\cos\theta = -0.50985$
   $\theta = \text{arc cos}(-0.50985) = 120.654°$

Looks about correct.

QUESTION 10:

What is the cosine of the angle between these unit vectors:

$\mathbf{s} = (1, 0, 1)^T / \sqrt{2}$
$\mathbf{t} = (1, 1, 1)^T / \sqrt{3}$

Click Here after you have answered the question

A good answer might be:

What is the cosine of the angle between these unit vectors:

$\mathbf{s} = (1, 0, 1)^T/\sqrt{2}$
$\mathbf{t} = (1, 1, 1)^T/\sqrt{3}$
$\cos\theta = (1, 0, 1)^T \cdot (1, 1, 1)^T / (\sqrt{2}\sqrt{3}) = 2/(\sqrt{2}\sqrt{3}) = \sqrt{2} / \sqrt{3} = 0.8164$

Geometry

The dot product formula can be used to work out some geometric problems that otherwise would be hard. In the following figure, the red edges of the figure coincide with the coordinate axes. They share a mutual endpoint at (0, 0, 0). The edge along the x axis ends at x=2; the edge along the y axis ends at y=3; and the edge along the z axis ends at z=4. The remaining edges connect these endpoints. Now say that you wanted to calculate the angle between the two green edges. This might be fairly tedious using trigonometry. But now you can:

1. Calculate a displacement vector for each green edge (by subtracting endpoints.)
2. Normalize each vector.
3. Use the dot product rule for the angle between them.

QUESTION 11:

What is the angle between the two green edges?

Click Here after you have answered the question

A good answer might be:

60.051°

Answer Worked Out

In loving detail:

1. The endpoints are:
   (0, 3, 0) to (2, 0, 0)
   (0, 3, 0) to (0, 0, 4)
2. The displacement vectors are:
   $\mathbf{a} = (2, 0, 0) - (0, 3, 0) = (2, -3, 0)^T$
   $\mathbf{b} = (0, 0, 4) - (0, 3, 0) = (0, -3, 4)^T$
3. The lengths are:
   $|\,\mathbf{a}\,|^2 = (2, -3, 0)^T \cdot (2, -3, 0)^T = 13$
   $|\,\mathbf{b}\,|^2 = (0, -3, 4)^T \cdot (0, -3, 4)^T = 25$
4. The normalized vectors are:
   $\mathbf{a_u} = (2, -3, 0)^T / \sqrt{13}$
   $\mathbf{b_u} = (0, -3, 4)^T / 5$
5. The dot product is:
   $\mathbf{a_u} \cdot \mathbf{b_u} = (2, -3, 0)^T \cdot (0, -3, 4)^T / (5\sqrt{13})$
   $= 9 / (5\sqrt{13}) = 0.49923$
6. The angle is:
   $\cos\theta = 0.49923$
   $\theta = \text{arc cos } 0.49923 = 60.051°$

QUESTION 12:

If you had calculated the displacement vectors in the opposite manner (by subtracting *from* (0, 3, 0)) would it have affected the answer?

Click Here after you have answered the question

A good answer might be:

No--the vectors would point in opposite directions, but their dot product would be the same.

End of the Chapter

This chapter did little but work with the formula for the cosine of the angle between two vectors. Only a few topics may need reviewing:

- The formula for the cosine of the angle between unit vectors.
- List of steps to take in applying it to non-unit vectors.
- Using vectors for problems in solid geometry.

Click on a term to see where it was discussed in this chapter. Remember to click on the "Back" arrow of your browser to get back to this page.

Click here to go back to the main menu.

You have reached the End.

created: 08/15/00; corrected: 10/28/00

CHAPTER 11 --- Projections

As discussed in chapter one, 3D computer graphics consists of two activities: (1) Creating an imaginary world inside a computer, and (2) producing two dimensional images of that world from various viewpoints.

Producing a 2D image from a 3D image is an example of **projection**. This chapter discusses the fundamental notions of projection. The advanced topics are left for your graphics text book.

Chapter Topics:

- *Projections in general.*
- *Projecting on vector onto another.*
- *Example vector projections in 2D.*
- *Example vector projections in 3D.*

QUESTION 1:

Is there a *unique* 2D image that can be created from a given 3D model?

Click Here after you have answered the question

Is there a *unique* 2D image that can be created from a give 3D model?

A good answer might be:

No, different projections of the 3D model give different 2D images.

Projections are not Unique

For example, the applet on the left projects a 3D model of a box onto the 2D view. Different positions of the viewpoint (use the slider) produce different 2D images. The 3D vectors of the model are being projected onto the viewing surface to form the 2D vectors of the image.

QUESTION 2:

Will any two vectors (not co-linear, and neither one the zero vector) lie in a unique plane?

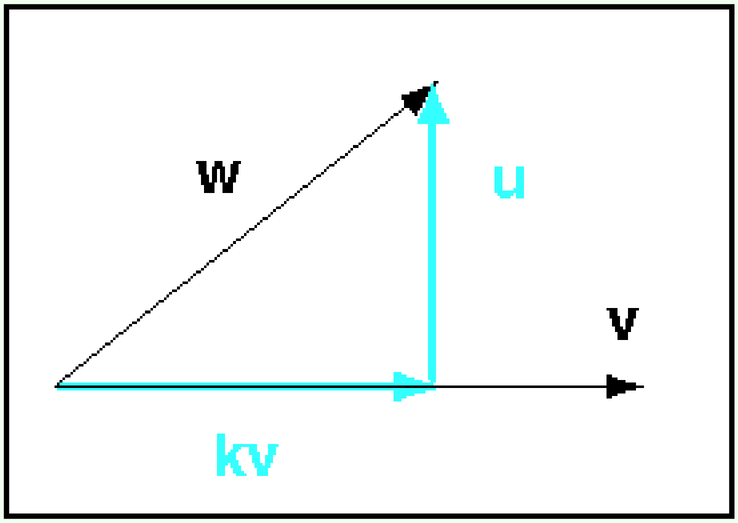
Click Here after you have answered the question

Will any two vectors (not co-linear) lie in a plane?

A good answer might be:

Yes.

Two Vectors Define a Plane



You may recall from high school geometry that two lines (not colinear) define a plane. This is the same thing. Most of the examples in this chapter will use 2D vectors. But you can think of the examples as 3D by regarding the plane as the one two 3D vectors determine.

The figure shows a vector **w** and an arbitrary vector **v**. For convenience in visualization, their tails start from the same point (but, of course, vectors have no position).

Also for convenience, the vector **v** is horizontal; but it could be at any orientation.

QUESTION 3:

Does the scaled vector k**v** (where k is a scalar) have the same orientation as **v**?

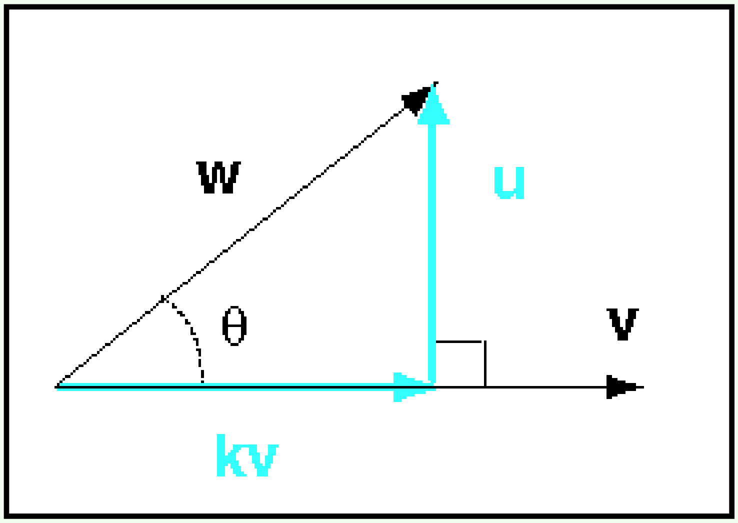
Click Here after you have answered the question

Does a scaled vector have the same orientation as the vector?

A good answer might be:

Yes---scaling changes the *length* of a vector, but not its orientation.

Projection of one Vector onto Another



In the diagram **w** and **v** are any two vectors. We want a vector **u** that is orthogonal to **v**. And we want scalar k so that:

$$\mathbf{w} = k\mathbf{v} + \mathbf{u}$$

Then k**v** is called the **projection** of **w** onto **v**.

QUESTION 4:

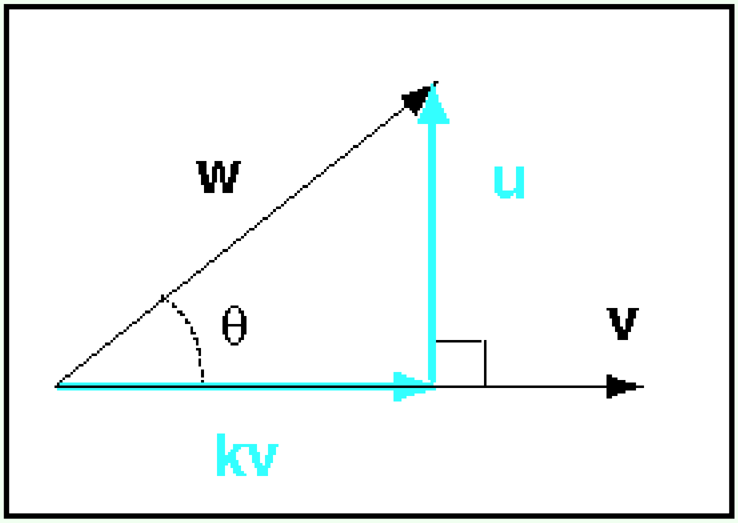
The vector **u** is orthogonal to **v**. What is true of **u** · **v**?

Click Here after you have answered the question

A good answer might be:

If **u** is orthogonal to **v**, then $\mathbf{u} \cdot \mathbf{v} = 0.0$

Solving for kv



We want: $\mathbf{w} = k\mathbf{v} + \mathbf{u}$, with the condition that **u** is orthogonal to **v**. Using trigonometry:

The length of k**v** = $|\mathbf{w}| \cos \theta$
The orientation of k**v** is $\mathbf{v_u}$

Remember that unit vectors show orientation in 3D space. The orientation of **v** is $\mathbf{v_u}$. (In the picture, this happens to be horizontal, but that is just for convenience). Since θ is $\mathbf{w_u} \cdot \mathbf{v_u}$, we have what we need:

$$k\mathbf{v} = |\mathbf{w}| (\mathbf{w_u} \cdot \mathbf{v_u}) \mathbf{v_u}$$

This formula may look awful, but it is not. This part:

$$|\mathbf{w}| (\mathbf{w_u} \cdot \mathbf{v_u})$$

is a scalar. It adjusts **v** to to required length. In the picture this is the length of the horizontal baby blue line.

The remaining part: $\mathbf{v_u}$, just says, "same orientation as **v**."

QUESTION 5:

(Review:) How do you compute a unit vector?

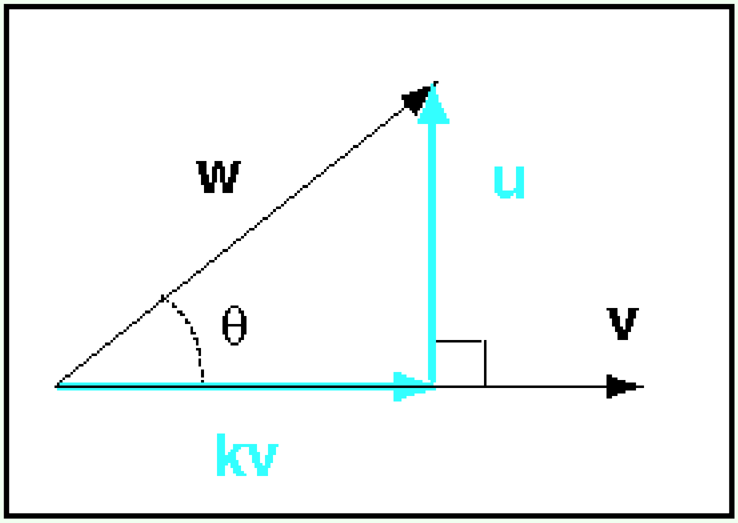
Click Here after you have answered the question

(Review:) How do you compute a unit vector?

A good answer might be:

Divide a vector by its length

Projection Procedure



Rather than memorize the formula for projection, follow what it means step-by-step. To project **w** onto **v**:

1. Compute the lengths:
   $| \mathbf{w} | = \sqrt{(\mathbf{w} \cdot \mathbf{w})}$
   $| \mathbf{v} | = \sqrt{(\mathbf{v} \cdot \mathbf{v})}$
2. Compute the unit vectors:
   $\mathbf{w_u} = \mathbf{w} / | \mathbf{w} |$
   $\mathbf{v_u} = \mathbf{v} / | \mathbf{v} |$
3. Compute the cosine of the angle between the vectors:
   $\mathbf{w_u} \cdot \mathbf{v_u}$
4. Assemble the projection:
   $k\mathbf{v} = | \mathbf{w} | (\mathbf{w_u} \cdot \mathbf{v_u}) \mathbf{v_u}$

QUESTION 6:

Is **w** projected onto **v** the same vector as **v** projected onto **w** ?

Click Here after you have answered the question

A good answer might be:

w projected onto **v** is **NOT** the same as **v** projected onto **w**

The vector resulting from a projection is oriented in the direction of *the vector projected onto*.

Finding u

After you have found k**v** it is easy to find the orthogonal vector, **u**:

w = k**v** + **u**,   so   **u** = **w** - k**v**

In the above formula, "+" means vector addition and "-" means vector subtraction.

So far in this chapter the vectors have all been geometrical vectors. There has been no mention of coordinate frames or column matrices.

QUESTION 7:

Will the results of this chapter work when vectors are represented with column matrices?

Click Here after you have answered the question

Will the results of this chapter work when vectors are represented with column matrices?

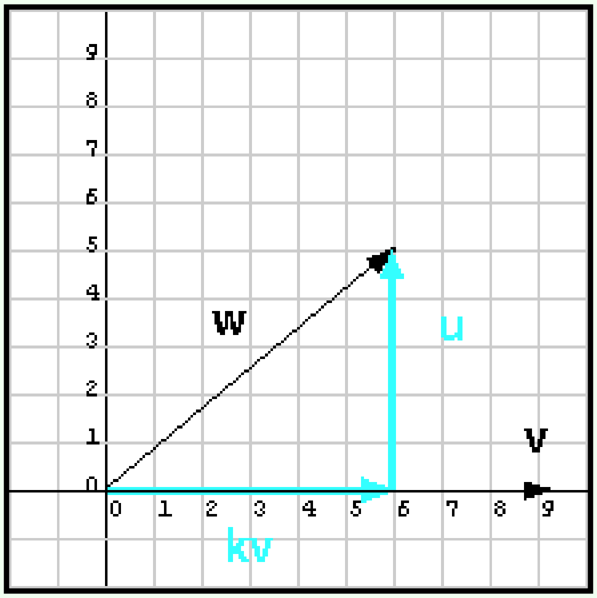
A good answer might be:

Thankfully, yes.

Example

Time for an example. In the diagram the answer can simply be read off the graph paper. But pretend you didn't notice that.

The vector **w** is represented by by $(6, 5)^T$. The vector **v** is represented by by $(9, 0)^T$. Find k**v** and **u**.



1. Compute the lengths:
   $| \mathbf{w} | = \sqrt{((6, 5)^T \cdot (6, 5)^T)} = 7.81$
   $| \mathbf{v} | = \sqrt{((9, 0)^T \cdot (9, 0)^T)} = 9$
2. Compute the unit vectors:
   $\mathbf{w_u} = (6, 5)^T / 7.81$
   $\mathbf{v_u} = (9, 0)^T / 9 = (1, 0)^T$
3. Compute the cosine of the angle between the vectors:
   $\mathbf{w_u} \cdot \mathbf{v_u} = (1/7.81)\,(6, 5)^T \cdot (1, 0)^T = 6/7.81$
4. Assemble the projection:
   $k\mathbf{v} = | \mathbf{w} |\,(\mathbf{w_u} \cdot \mathbf{v_u})\,\mathbf{v_u}$
   $k\mathbf{v} = 7.81\,(6/7.81)\,(1, 0)^T = 6(1, 0)^T = (6, 0)^T$
5. Compute the orthogonal vector:
   $\mathbf{u} = \mathbf{w} - k\mathbf{v}$
   $\mathbf{u} = (6, 5)^T - (6, 0)^T = (0, 5)^T$

The result is that **u** = $(6, 0)^T + (0, 5)^T$, as expected. Of course, the example was easy.

QUESTION 8:

Was it actually necessary to compute **| w |**?

Click Here after you have answered the question

Was it actually necessary to compute **| w |**?

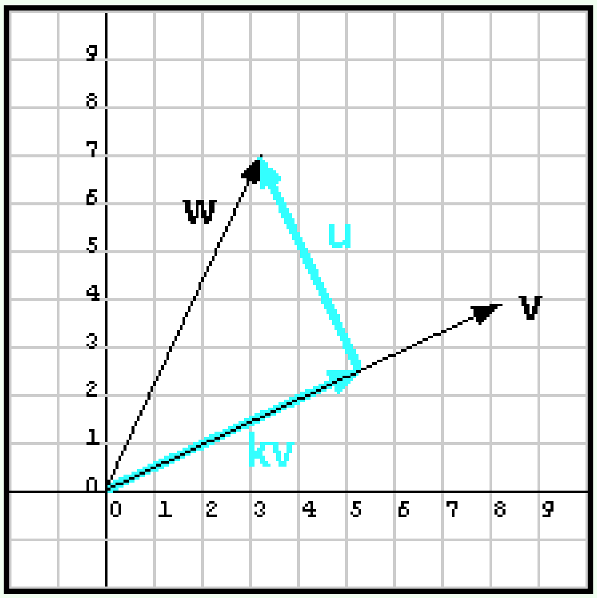
A good answer might be:

No. It cancelled out of the final result.

Another Example

You don't have to compute the length of **w**, **| w |**. It cancels out before the final answer. And, all you need is the square of the length of **v**, $|\mathbf{v}|^2$. Some books show formulae for projection that make use of these facts (but, to my taste, are less intuitive).

The diagram shows another example, this time not so easy. The vector **w** is represented by by $(3.2, 7)^T$. The vector **v** is represented by by $(8, 4)^T$. Find k**v** and **u**.



1. Compute the lengths:
   $|\mathbf{w}|$ = (keep it symbolic)
   $|\mathbf{v}|^2 = ((8, 4)^T \cdot (8, 4)^T) = 80$
2. Compute the unit vectors:
   $\mathbf{w_u} = (3.2, 7)^T / |\mathbf{w}|$
   $\mathbf{v_u} = (8, 4)^T / |\mathbf{v}|$
3. Compute the cosine of the angle between the vectors:
   $\mathbf{w_u} \cdot \mathbf{v_u} = (3.2, 7)^T / |\mathbf{w}| \cdot (8, 4)^T / |\mathbf{v}|$
   $= 53.6/(|\mathbf{w}||\mathbf{v}|)$
4. Assemble the projection:
   $k\mathbf{v} = |\mathbf{w}| (\mathbf{w_u} \cdot \mathbf{v_u}) \mathbf{v_u}$
   $k\mathbf{v} = |\mathbf{w}| [53.6 / (|\mathbf{w}||\mathbf{v}|)] (8, 4)^T / |\mathbf{v}|$
   $k\mathbf{v} = 53.6 / (|\mathbf{v}|) (8, 4)^T / |\mathbf{v}|$
   $k\mathbf{v} = 53.6 / (|\mathbf{v}|^2) (8, 4)^T$
   $k\mathbf{v} = 53.6 / 80 (8, 4)^T$
   $k\mathbf{v} = ((53.6*8)/80, (53.6*4)/80)^T = (5.3, 2.68)^T$
5. Compute the orthogonal vector:
   $\mathbf{u} = \mathbf{w} - k\mathbf{v}$

$\mathbf{u} = (3.2, 7)^T - (5.3, 2.68)^T = (-2.1, 4.32)^T$

QUESTION 9:

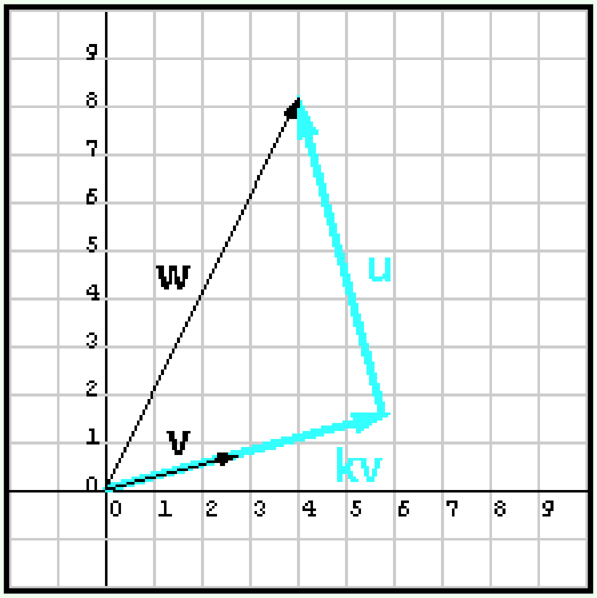
Is $\mathbf{w} = k\mathbf{v} + \mathbf{u}$, as it should?

Click Here after you have answered the question

A good answer might be:

$(5.3, 2.68)^T + (-2.1, 4.32)^T = (3.2, 7.0)^T$, as expected.

The Point of All This



The point of all this is for you to get a gut-level understanding of what is going on when one vector is projected onto another. This, as previously mentioned, is constantly occuring when a computer is displaying graphics. Much of the calculation takes place in the computer's graphics hardware. Millions of vectors are projected per second. If you want to know what is going on with computer graphics, this is something you need to know.

But you are not going to make your living actually doing the calculations.† And I have a headache from setting all those equations in HTML. So let's skip the math and just read off the answer from the diagram.

†A friend of mine does make a living doing projections. But he does it from a projection booth in a movie theater, and even that job is being taken over by computers.

QUESTION 10:

w = (_____, _____)^T **v** = (_____, _____)^T

k**v** = (_____, _____)^T **u** = (_____, _____)^T

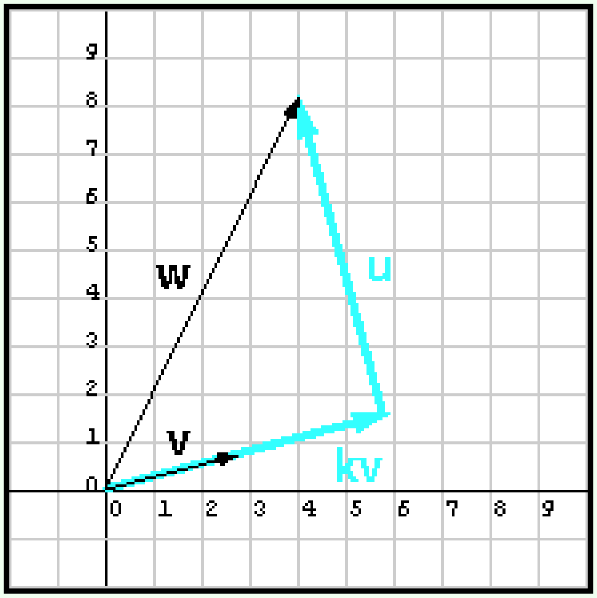
Click Here after you have answered the question

A good answer might be:

$\mathbf{w} = (4, 8.2)^T \quad \mathbf{v} = (2.8, 1.75)^T$

$k\mathbf{v} = (5.8, 1.6)^T \quad \mathbf{u} = (-1.8, 6.4)^T$

Check the Answer



The above answers look good to me (especially since I drew the picture), but let us see if they are reasonable.

First: Does $\mathbf{w} = k\mathbf{v} + \mathbf{u}$?

$(5.8, 1.6)^T + (-1.8, 6.4)^T = (4, 8.0)^T = \mathbf{w}$ (well, almost)

Next: Does $\mathbf{v} \cdot \mathbf{u} = 0.0$?

$(5.8, 1.6)^T \cdot (-1.8, 6.4)^T = -10.44 + 10.24 = -0.2 = 0.0$ (nearly)

QUESTION 11:

Think about all possible vectors **w** and **v** and the projection of **w** onto **v**.

1. Is it possible for the vector k**v** to be longer than **v**?
2. Is it possible for the vector k**v** to be longer than **w**?

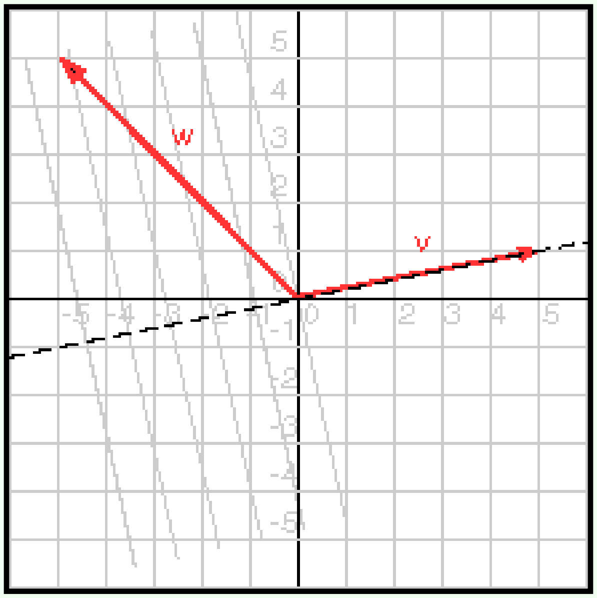
Click Here after you have answered the question

A good answer might be:

1. Is it possible for the vector k**v** to be longer than **v**?
   Yes, as in the previous example.
2. Is it possible for the vector k**v** to be longer than **w**?
   No, the longest k**v** can be is **| w |**, when **w** and **v** are co-linear.

Groan! Another Example

will they never end.....



Notice that the projection of **w** onto **v** does NOT depend on the length of **v**. Only the direction of vector **v** affects the results. The length of the projection is **| w |** cos θ, where θ is the angle between the vectors. Since the cosine function is never more than one, the length of the projection will never be longer than the projected vector.

(These are not facts to be memorized; the idea is to practice visualizing what is going on by thinking about such things). And nothing better for visualization than other example.

QUESTION 12:

To make this example especially hard, I've changed the colors. Project **w** onto **v** by examining the diagram.

w = (_____, _____)^T **v** = (_____, _____)^T
k**v** = (_____, _____)^T **u** = (_____, _____)^T

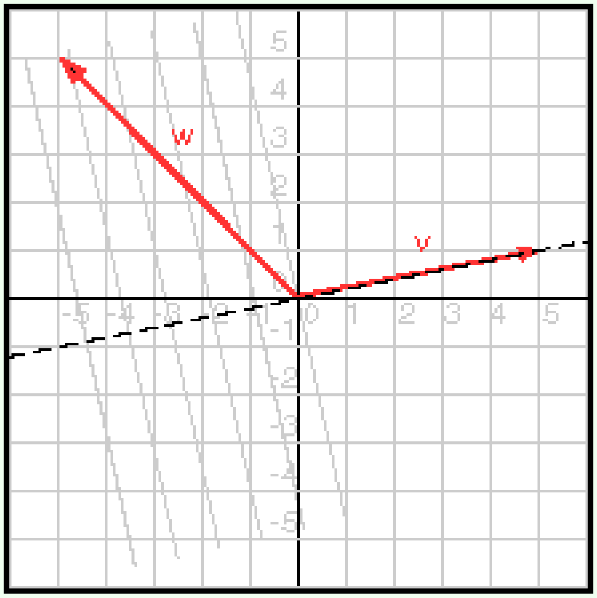
Click Here after you have answered the question

A good answer might be:

$\mathbf{w} = (-5, 5)^T \qquad \mathbf{v} = (5, 1)^T$

$k\mathbf{v} = (-4.2, -.85)^T \qquad \mathbf{u} = (-1.3, 5.8)^T$

Eyeball Estimate



Those answers were done by reading off the picture. No guarantee that they are accurate. So let us check:

First: Does $\mathbf{w} = k\mathbf{v} + \mathbf{u}$?

$$(-4.2, -0.85)^T + (-1.3, 5.8)^T = (-5.5, 4.95)^T$$
$$= \mathbf{w} \quad \text{(almost)}$$

Next: Does $\mathbf{v} \cdot \mathbf{u} = 0.0$?

$$(5, 1)^T \cdot (-1.3, 5.8)^T = -6.5 + 5.8$$
$$= -0.7 = 0.0 \quad \text{(almost)}$$

QUESTION 13:

Can the projection of **w** onto **v** point in the *opposite* direction of **v** ?

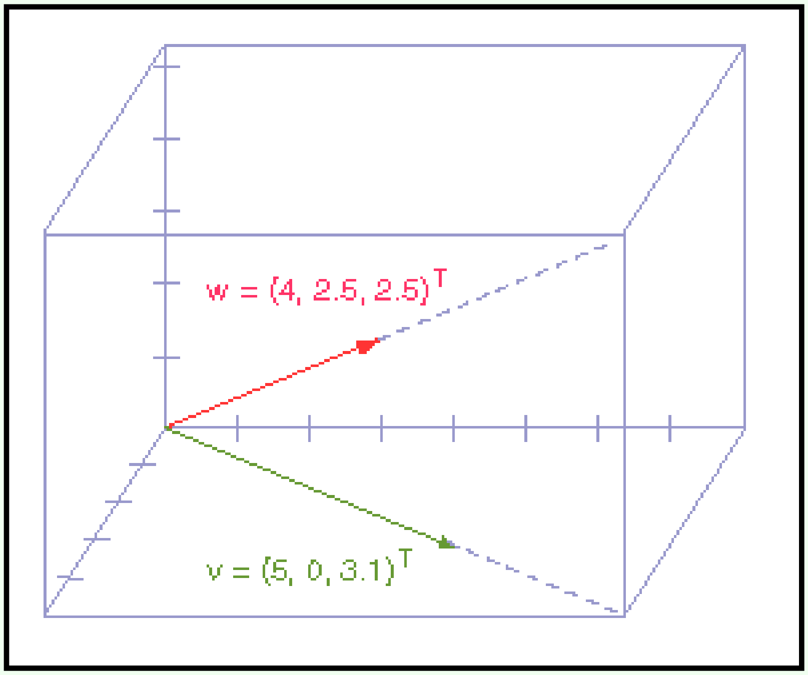
Click Here after you have answered the question

Can the projection of **w** onto **v** point in the *opposite* direction of **v** ?

A good answer might be:

Yes, as it did in that example.

Example in Three Dimensions



The diagram shows **w** = $(4, 2.5, 2.5)^T$ (red vector) and **v** = $(5, 0, 3.1)^T$ (green vector).

QUESTION 14:

Mentally project **w** onto **v**. (Hint: remember that the length of **v** does not matter; only its orientation. So mentally "drop" the arrowhead of **w** onto the dotted line).

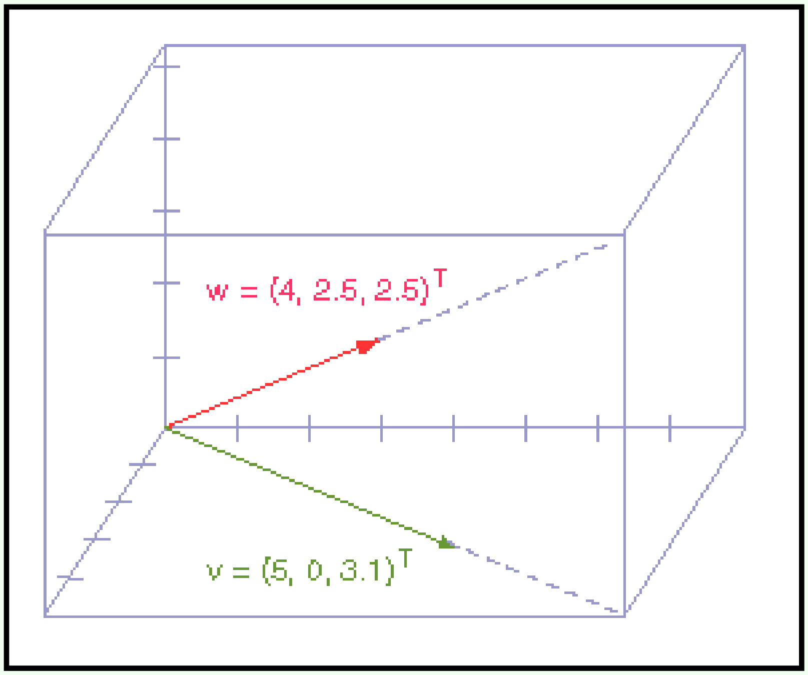
Click Here after you have answered the question

A good answer might be:

The projection of **w** = $(4, 2.5, 2.5)^T$ onto **v** = $(5, 0, 3.1)^T$ is k**v** = $(4, 0, 2.5)^T$

Collapsing one Dimension

In this case, projecting **w** onto **v** involved "collapsing" the y dimension of **w**. Lets see if the same result comes out mathematically. The same steps work for 3D as for 2D:



1. Compute the lengths:
   $|\mathbf{w}|$ = (keep it symbolic)
   $|\mathbf{v}|^2 = ((5, 0, 3.1)^T \cdot (5, 0, 3.1)^T) = 34.61$
2. Compute the unit vectors:
   $\mathbf{w_u} = (4, 2.5, 2.5)^T / |\mathbf{w}|$
   $\mathbf{v_u} = (5, 0, 3.1)^T / |\mathbf{v}|$
3. Compute the cosine of the angle between the vectors:
   $\mathbf{w_u} \cdot \mathbf{v_u}$
   $= (4, 2.5, 2.5)^T / |\mathbf{w}| \cdot (5, 0, 3.1)^T / |\mathbf{v}|$
   $= 27.75/(|\mathbf{w}||\mathbf{v}|)$
4. Assemble the projection:
   $k\mathbf{v} = |\mathbf{w}| (\mathbf{w_u}\cdot\mathbf{v_u})\, \mathbf{v_u}$
   $k\mathbf{v} = |\mathbf{w}| (27.75 / (|\mathbf{w}||\mathbf{v}|))\ (5, 0, 3.1)^T / |\mathbf{v}|$
   $k\mathbf{v} = (27.75 / (|\mathbf{v}|))\ ((5, 0, 3.1)^T / |\mathbf{v}|)$
   $k\mathbf{v} = (27.75 / |\mathbf{v}|^2)\ (5, 0, 3.1)^T$
   $k\mathbf{v} = (27.75 / 34.61)\ (5, 0, 3.1)^T$
   $k\mathbf{v} = ((27.75*5)/34.61, 0, (27.75*3.1)/34.61^T$
   $k\mathbf{v} = (4.00, 0, 2.49)^T$

So the visual estimate k**v** = $(4, 0, 2.5)^T$ matches the mathematical result, $(4.00, 0, 2.49)^T$. The orthogonal vector can easily be computed:

$$\mathbf{u} = \mathbf{w} - k\mathbf{v}$$
$$\mathbf{u} = (4, 2.5, 2.5)^T - (4, 0, 2.5)^T = (0, 2.5, 0)^T$$

QUESTION 15:

Keep the same vector **w** as in the diagram. Visualize projecting **w** onto various vectors **v'**. Can you think of a vector **v'** that results in a projection k**v'** that collapses both the y and z dimensions of **w** ?

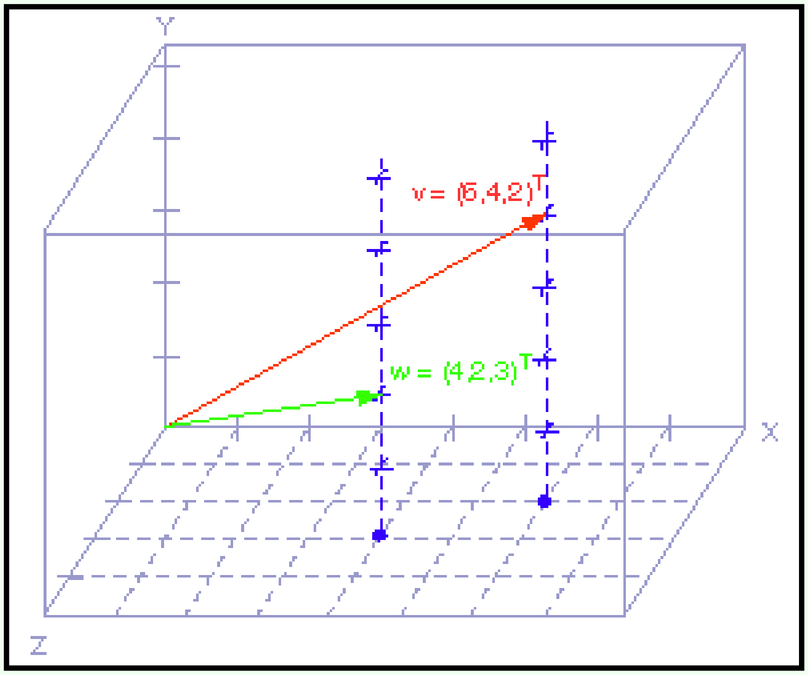
Click Here after you have answered the question

Keep the same vector **w** as in the diagram. Visualize projecting **w** onto various vectors **v'**. Can you think of a vector **v'** that results in a projection k**v'** that collapses both the y and z dimensions of **w** ?

A good answer might be:

It is probably harder to read the question than to figure out the result. If **w** = $(4, 2.5, 2.5)^T$ is projected onto **v'** = $(1, 0, 0)^T$ the result is k**v** = $(4, 0, 0)^T$

Another Example



Of course, that is what coordinate axes are: when a vector in 3D space is projected onto a coordinate axes you get one of the three dimensions of that vector. So let us look at an example that is not quite so easy:

In the picture **w** = $(4, 2, 3)^T$ and
v = $(6, 4, 2)^T$.

QUESTION 16:

Visually project **w** onto **v**.

Click Here after you have answered the question

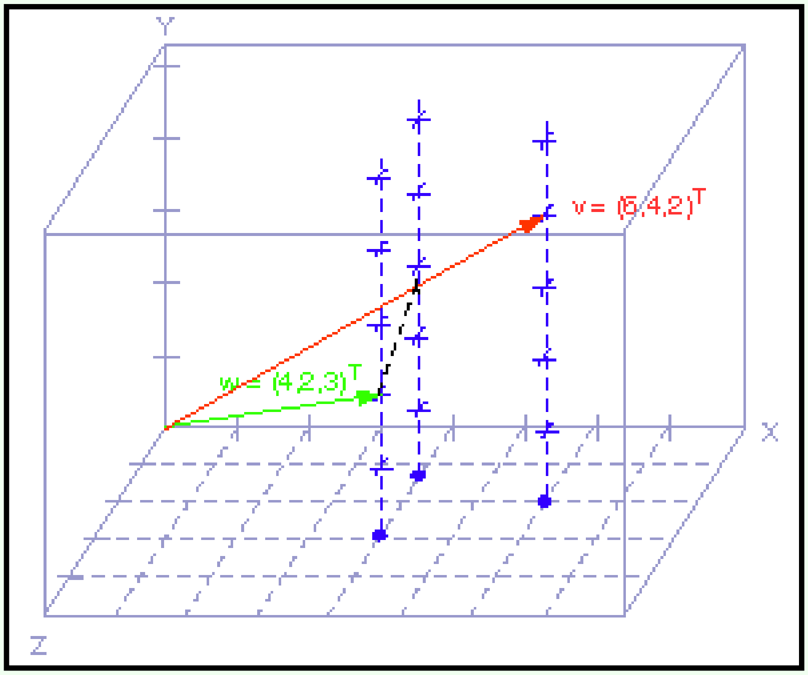
A good answer might be:

My visual estimate is that if you "drop" the arrow head of **w** along a perpendicular to **v**, you hit **v** at about 4/10 the length of **v**. So k is about 0.4 and k**v** $= 0.4\,(6, 4, 2)^T = (2.4, 1.6, 0.8)^T$

It turns out that I was grossly wrong. Hope you did better.

Checking the Answer

The visual approach to 3D problems is hard, unless you actually have a 3D image in front of you. The correct answer calls for some figuring:



1. Compute the lengths:
   $|\mathbf{w}| =$ (keep it symbolic)
   $|\mathbf{v}|^2 = ((6, 4, 2)^T \cdot (6, 4, 2)^T)$
   $= 56$
2. Compute the unit vectors:
   $\mathbf{w_u} = (4, 2, 3)^T / |\mathbf{w}|$
   $\mathbf{v_u} = (6, 4, 2)^T / |\mathbf{v}|$
3. Compute the cosine of the angle between the vectors:
   $\mathbf{w_u} \cdot \mathbf{v_u} = (4, 2, 3)^T \cdot (6, 4, 2)^T / |\mathbf{w}||\mathbf{v}|$
   $= 38/(|\mathbf{w}|\ |\mathbf{v}|)$
4. Assemble the projection:
   $k\mathbf{v} = |\mathbf{w}|\ (\mathbf{w_u} \cdot \mathbf{v_u})\ \mathbf{v_u}$
   $k\mathbf{v} = |\mathbf{w}|\ \{38 / (|\mathbf{w}||\mathbf{v}|)\}\ \{(6, 4, 2)^T / |\mathbf{v}|\}$
   $k\mathbf{v} = \{38 /|\mathbf{v}|\}\ \ \{(6, 4, 2)^T / |\mathbf{v}|\}$
   $k\mathbf{v} = \{38 /|\mathbf{v}|^2\}\ \ (6, 4, 2)^T$
   $k\mathbf{v} = \{38/56\}\ \ (6, 4, 2)^T$
   $k\mathbf{v} = 0.679\ \ (6, 4, 2)^T$
   $k\mathbf{v} = (\ 4.07, 2.7, 1.36)^T$

QUESTION 17:

What is **u**, where **u** + k**v** = **w**? (In the diagram **u** is the dotted black line).

Click Here after you have answered the question

A good answer might be:

$\mathbf{u} + k\mathbf{v} = \mathbf{w}$, so $\mathbf{u} = \mathbf{w} - k\mathbf{v} = (4, 2, 3)^T - (4.07, 2.7, 1.36)^T = (-0.07, -0.7, 1.64)^T$

End of the Chapter

You have reached the end this chapter. Your next project is to review the new terms encountered in these pages:

- Two vectors define a plane.
- Projection of one vector onto another.
- Formula for vector projection.
- Step-by-step procedure for vector projection.
- Orthogonal component of vector projection.
- Simplified calculation.

The projected future is really grim. More of this stuff. You are sure to get cross.

Click here to go back to the main menu.

You have reached the End.

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CHAPTER 12 --- Vector Cross Product

The vector cross product is used often in explaining things, but not very often in actual computation. It does not have the mathematical status nor the practical importance of the other operations we have seen.

Chapter Topics:

The most important use of the cross product in computer graphics is to find a vector perpendicular to a plane.

QUESTION 1:

(Review:) How many vectors are needed to define a plane?

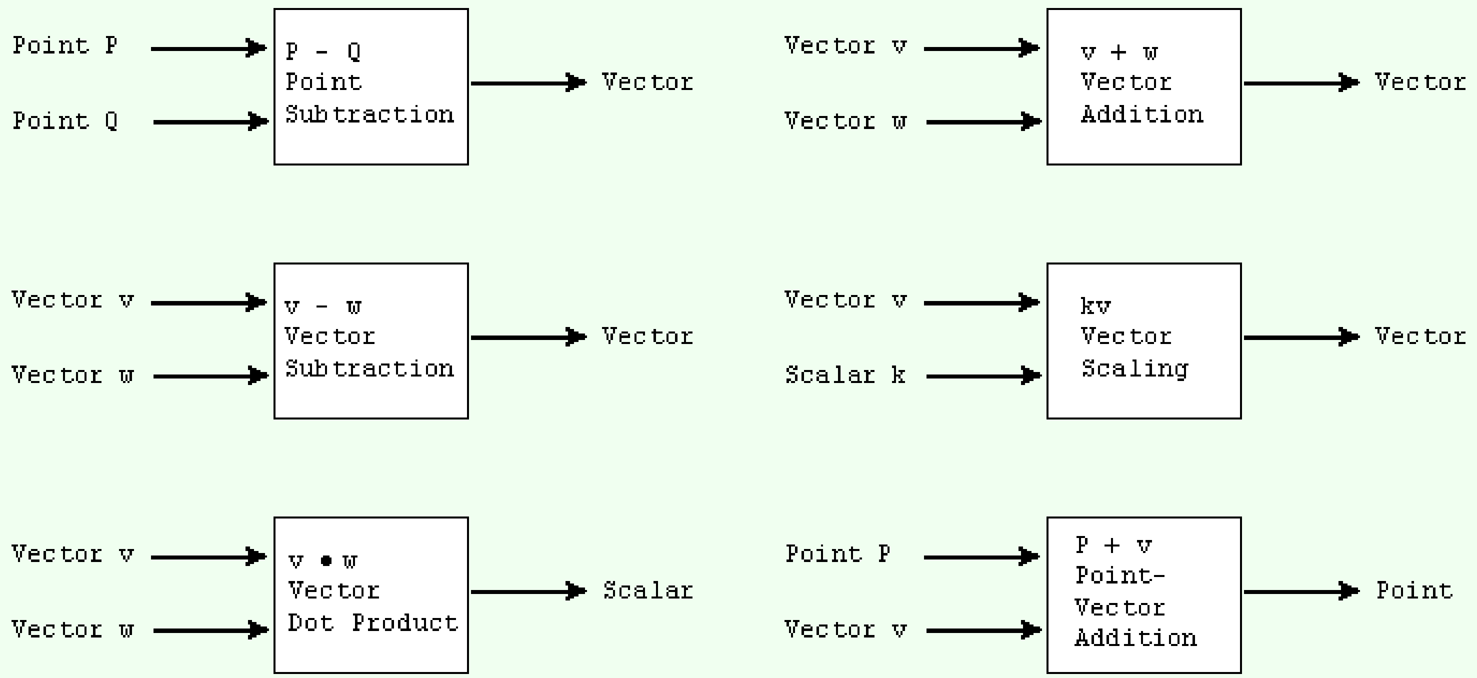
Click Here after you have answered the question

A good answer might be:

Two vectors, not co-linear.

Operands, Operators, and Results

So far these notes have discussed the following:



QUESTION 2:

In the above, is there a multiplication-like operation that takes two vector operands and outputs a vector result?

Click Here after you have answered the question

A good answer might be:

No.

Definition of the Cross Product

The vector cross product takes two vector operands to produce a vector result. The result, like all geometric vectors, has two properties: length and orientation.

If **u** and **v** are vectors in three dimensional space (only), then **u** × **v** is a three dimensional vector, where:

> Length:
>
> > $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$, where θ is the angle between **u** and **v**.
>
> Orientation:
>
> > **u** × **v** is perpendicular to both **u** and **v**. The choice (out of two) orientations perpendicular to **u** and **v** is made by the *right hand rule.*

To find a vector perpendicular to a particular plane, compute the cross product of two vectors in that plane. But there are two directions perpendicular to the plane. Which one does the cross product give you? That is determined by the *right hand rule*, which will be explained shortly.

QUESTION 3:

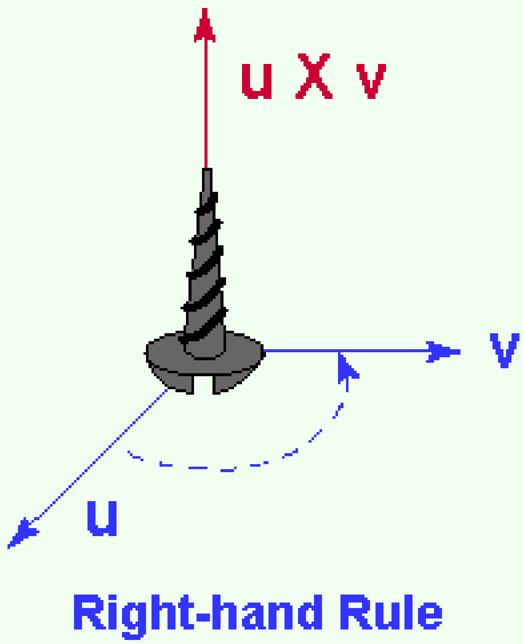
A wood screw is positioned perpendicular to a plank of wood. You wish drive the screw into the wood with a screw driver. Which direction must you twist the screw driver?

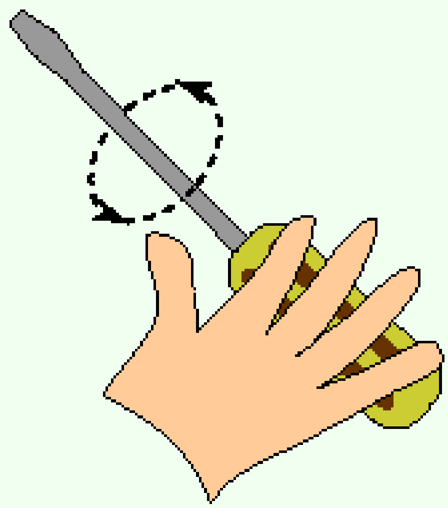
Click Here after you have answered the question

A good answer might be:

A right-handed screw (the normal kind) must be turned clockwise into the wood (viewed from its head).

Carpentry of Vectors





The cross pro
vectors is perp
both; the right
picks the one
possible perpe
directions.

The diagrams
hand rule. The
operands **is in**
result of **u** ×
the same direc
must point if t
u to **v** drives i

Or, think of th
your thumb po
fingers curl ar
v.

QUESTION 4:

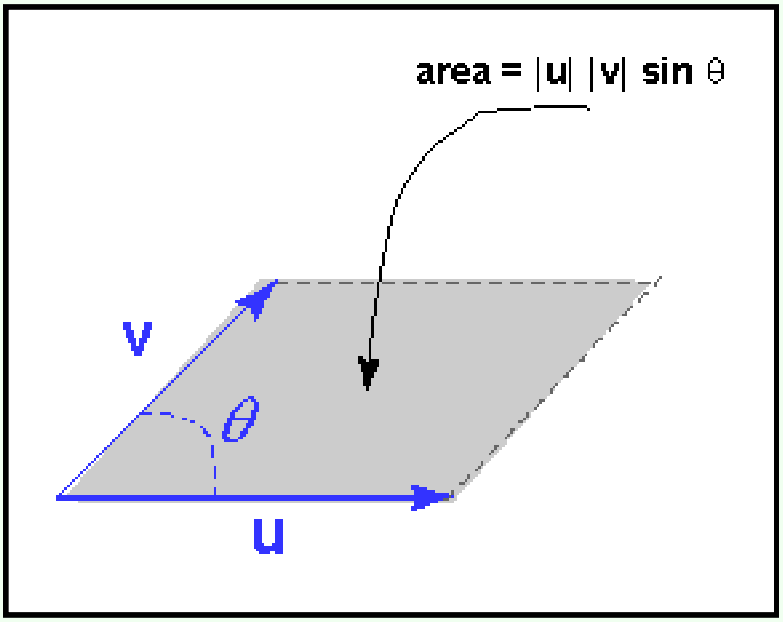
Does **u** × **v** point in the same direction as **v** × **u** ? (Hint: get out your screw driver).

[Click Here after you have answered the question]

A good answer might be:

No. The two possible cross products between **u** and **v** *point in opposite directions.*

Visualizing the Cross Product



The picture shows two vectors **u** and **v** and the parallelogram they define between them.

The area of the parallelogram is: | **u** | | **v** | sin θ, where θ is the enclosed area.

This area is the same as the magnitude of the cross product. This fact is sometimes helpful in visualizing the cross product. For example, if **u** is rotated so that its orientation approaches that of **v** (but its length is not changed), the area of the parallelogram gets close to zero, as does the magnitude of the cross product.

QUESTION 5:

What angle between **u** and **v** will *maximize* their cross product? (Hint: look at the diagram)

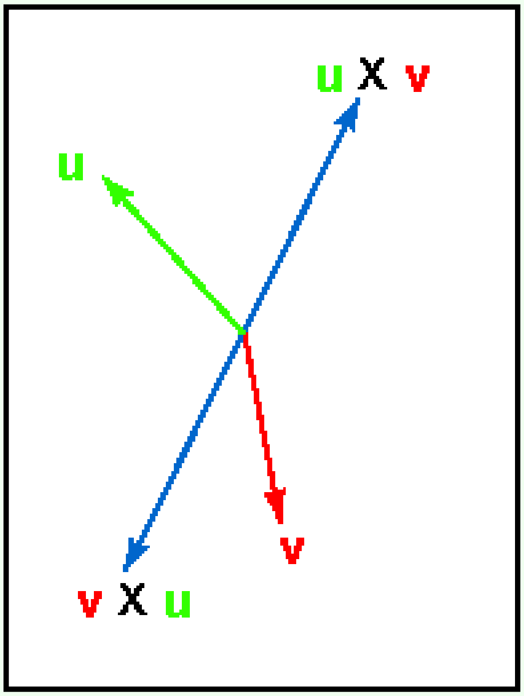
Click Here after you have answered the question

What angle between **u** and **v** will *maximize* their cross product?

A good answer might be:

90°

Reversing the Order of Operands



Review: To reverse the direction of a vector **w**, scale it by -1. In other words, **w** and **-w** are colinear but point in opposite directions. The diagram shows two cross products, their orientation determined by the right hand rule.

The magnitude of **u** × **v** is | **u** | | **v** | sin θ.

The magnitude of **v** × **u** is | **v** | | **u** | sin θ = | **u** | | **v** | sin θ.

So, **u** × **v** = -(**v** × **u**) .

QUESTION 6:

Does -(**v** × **u**) = **-v** × **u** ? (Decide this by looking at the diagram and visualizing both sides of the equation).

Click Here after you have answered the question

A good answer might be:

Yes, $-(\mathbf{v} \times \mathbf{u}) = -\mathbf{v} \times \mathbf{u}$

Colinear Operands

The length of $\mathbf{u} \times \mathbf{v}$ is $|\,\mathbf{u}\,|\,|\,\mathbf{v}\,|\sin\theta$. If **u** and **v** are colinear (parallel) what is the length of their cross product?

Since $\sin\theta = 0$ when $\theta = 0$, and the angle between colinear vectors is zero, the magnitude of the result is zero. The result is still a vector; it is the *zero vector* **0**.

$$\mathbf{u} \times \mathbf{u} = \mathbf{0}.$$

Also, since k**u** is colinear to **u** (for a scalar k), then:

$$(k\mathbf{u}) \times \mathbf{u} = \mathbf{0}.$$

QUESTION 7:

Is the result of $\mathbf{u} \times \mathbf{u}$ perpendicular to both operands?

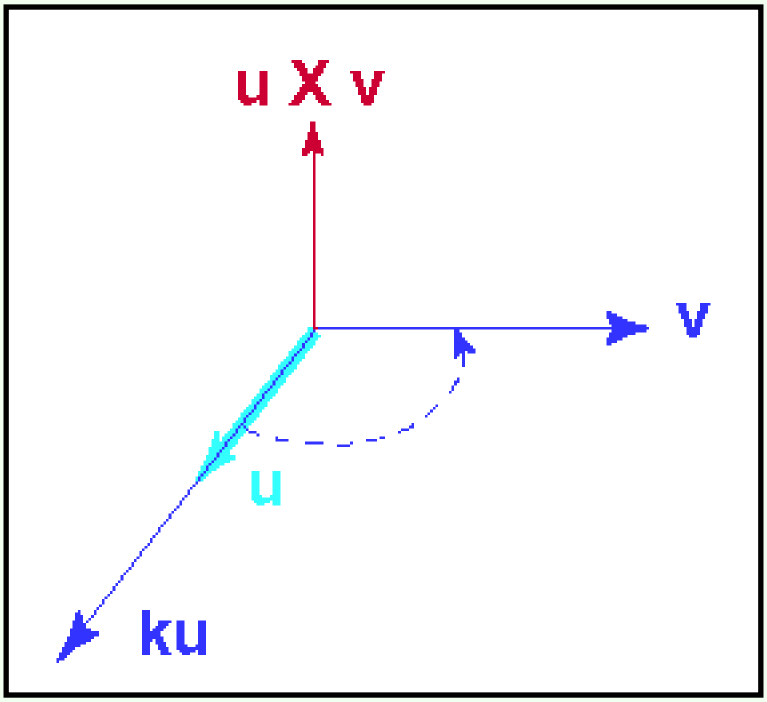
Click Here after you have answered the question

Is the result of $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ perpendicular to both operands?

A good answer might be:

Yes, because $\mathbf{0} \cdot \mathbf{u} = 0$, which means that they are perpendicular.

More Math Fun



What about $\mathbf{u} \times \mathbf{0}$? The magnitude $|\mathbf{u}| |\mathbf{0}| \sin\theta$ is zero, and the result is a vector, so:

$$\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$$

Now, consider $(k\mathbf{u}) \times \mathbf{v}$. Plug into the formula for the magnitude ($|\mathbf{u}| |\mathbf{v}| \sin\theta$) and for the direction (perpendicular to both operands).

QUESTION 8:

What is $(k\mathbf{u}) \times \mathbf{v}$?

Click Here after you have answered the question

A good answer might be:

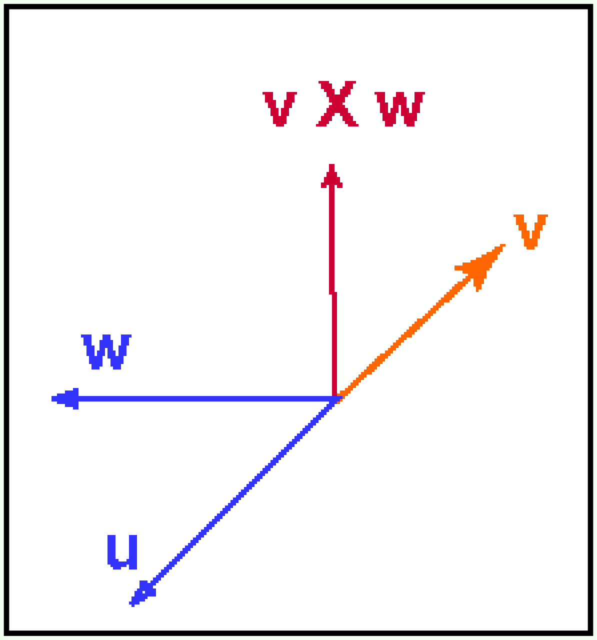
By examining the diagram, or working with the formula: (k**u**) × **v** = k(**u** × **v**)

Not Associative

Fussing with math gives the same result:

$$|(k\mathbf{u}) \times \mathbf{v}| = |k\mathbf{u}|\,|\mathbf{v}| \sin\theta = |k|\,|\mathbf{u}|\,|\mathbf{v}| \sin\theta$$

The magnitude is | k | times the magnitude of **u** × **v**. And the orientation of the result must be the same. So the answer is correct.



Another fact is that, in general, (**u** × **v**) × **w** =/= **u** × (**v** × **w**)

To see this, look at the diagram and mentally form the cross product (**u** × **v**) , then take the cross product of that with **w**. Then form the cross product (**v** × **w**) and take the cross procut of **u** with that.

QUESTION 9:

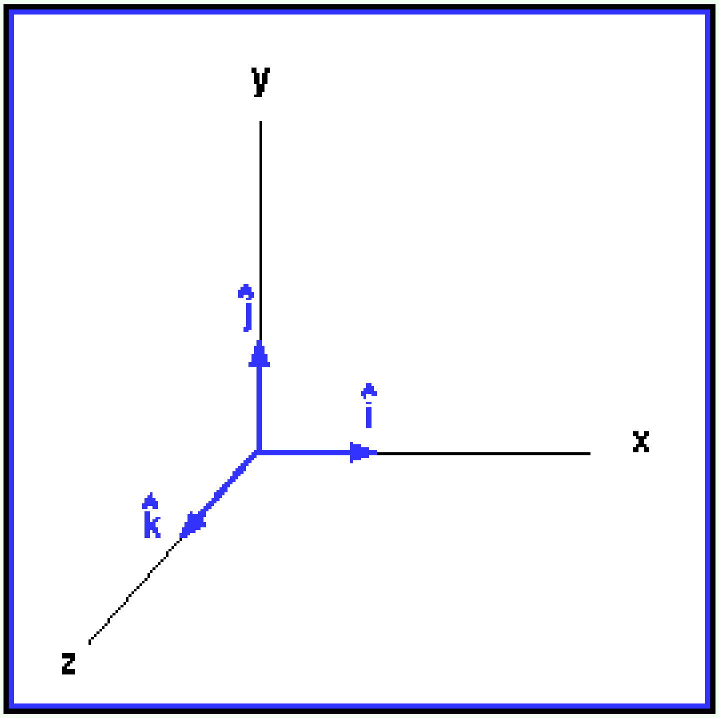
What is (**u** × **v**) in the diagram ?

Click Here after you have answered the question

A good answer might be:

0

More Properties



Another property is $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$. To see this, plug into the definition of the cross product.

Often the unit vectors in the direction of the x, y, and z axes (in whatever coordinate frame is currently being used) are called **i**, **j**, and **k**. Sometimes (as in the picture), little hats are put on each letter.

Since the length of each vector is one, and they are mutually perpendicular (so the sine of the angle between any two is one), the following is true:

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$
$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
$\mathbf{k} \times \mathbf{i} = \mathbf{j}$	$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

Rather than memorize this (probablity of success = 0), just draw the diagram and use the right hand rule.

QUESTION 10:

What is each of the following:

$\mathbf{i} \times \mathbf{i} =$
$\mathbf{j} \times \mathbf{j} =$
$\mathbf{k} \times \mathbf{k} =$

Click Here after you have answered the question

A good answer might be:

$\mathbf{i} \times \mathbf{i} = \mathbf{0}$
$\mathbf{j} \times \mathbf{j} = \mathbf{0}$
$\mathbf{k} \times \mathbf{k} = \mathbf{0}$

Computing Cross Product from Column Matrices

Say that the vector **u** is represented by $\mathbf{u} = (u_i, u_j, u_k)^T$
and that the vector **v** is represented by $\mathbf{v} = (v_i, v_j, v_k)^T$.

Then $\mathbf{u} \times \mathbf{v} = (u_j v_k - u_k v_j ,\ u_k v_i - u_i v_k ,\ u_i v_j - u_j v_i)^T$.

There is a pattern in how this is formed. Look at it a bit. But don't even think of memorizing it.

QUESTION 11:

What is: $(1, 2, 3)^T \times (0, 0, 0)^T$?

Click Here after you have answered the question

What is: $(1, 2, 3)^T \times (0, 0, 0)^T$?

A good answer might be:

Obviously $(1, 2, 3)^T \times \mathbf{0} = \mathbf{0}$

Or, plug into the formula $\mathbf{u} \times \mathbf{v} = (u_j v_k - u_k v_j ,\ u_k v_i - u_i v_k ,\ u_i v_j - u_j v_i)^T$.

$(1, 2, 3)^T \times (0, 0, 0)^T = (2\times0 - 3\times0,\ 3\times0 - 1\times0,\ 1\times0 - 2\times0)^T = \mathbf{0}$

Memory Aid

Here is a way to compute the cross product by arranging the elements of each vector into a determinant. The top row contains the symbols that stand for each axes. This is **not** the definition of cross product. In fact, it is not even a determinant. It is merely a memory aid.

$$\mathbf{u} = \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} \qquad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_i & u_j & u_k \\ v_i & v_j & v_k \end{vmatrix}$$

This only works for three dimensional vectors (recall the cross product not defined for vectors of any but three dimensions). Be careful to put the first vector's components into the second row, and the second vector's components into the third row.

QUESTION 12:

What is the cross product of $(1, 2, 1)^T$ with $(0, -1, 2)^T$? Fill in the blanks:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{vmatrix}$$

Click Here after you have answered the question

What is the cross product of $(1, 2, 1)^T$ with $(0, -1, 2)^T$?

A good answer might be:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$$

Evaluating the Determinant-like Thing

To evaluate this, find the cofactors of **i**, **j**, **k** in the top row. In the picture, the cofactors are circled in blue. The cofactor for **i**, for instance, is the elements of rows 2 and 3 other with the i^{th} column crossed out.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= \mathbf{i}\,(2 \times 2 - 1 \times (-1)) - \mathbf{j}\,(1 \times 2 - 1 \times 0) + \mathbf{k}\,(1 \times (-1) - 2 \times 0)$$

$$= 5\mathbf{i} - 2\mathbf{j} - 1\mathbf{k} = (5, -2, -1)^T$$

To evaluate a cofactor, multiply the two elements on the main diagonal, subtract from that the product of the two elements on the other diagonal. For example, for the cofactor of **i** this is $2 \times 2 - 1 \times (-1) = 2 + 1 = 5$

QUESTION 13:

What is the cross product of $(0, -1, 2)^T$ with $(1, 2, 1)^T$? Fill in the blanks. (NOTE: the order of the vectors has been reversed).

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{vmatrix}$$

Click Here after you have answered the question

A good answer might be:

What is the cross product of $(0, -1, 2)^T$ with $(1, 2, 1)^T$?

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

Another Evaluation

Evaluation follows the same pattern as before, but the second and third rows have been swapped.

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{vmatrix} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \mathbf{i}\,(-1 \times 1 - 2 \times 2) - \mathbf{j}\,(0 \times 1 - 2 \times 1) + \mathbf{k}(0 \times 2 - (-1) \times 1)$$

$$= -5\mathbf{i} + 2\mathbf{j} + 1\mathbf{k} = (-5, 2, 1)^T$$

This is the negative of the previous result, demonstrating that

$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

(You saw this before, only done with geometrical vectors).

QUESTION 14:

What does the determinant-like thing look like when you take the cross product of a vector by itself?

Click Here after you have answered the question

What does the determinant-like thing look like when you take the cross product of a vector by itself?

A good answer might be:

Rows two and three will be the same.

Two Rows Equal

Recall from the dark ages of high school math that if two rows of a determinant are the same, then it evaluates to zero. Each co-factor will evaluate to zero, resulting in $0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$.

The same will happen if one row is a multiple of another. These results reflect what we have already seen with geometrical vectors: $k\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

The other properties of the cross product of geometrical vectors are also true of the cross product of their column matrix representations.

QUESTION 15:

What were those properties again?

Click Here after you have answered the question

A good answer might be:

Here is a list:

End of Chapter

Right hand rule.

The orientation of $\mathbf{u}\times\mathbf{v}$ is in the "thumb" direction if your fingers curl from **u to v**.

Magnitude of Cross product.

The magnitude of $\mathbf{u}\times\mathbf{v}$ is the area of the parallelogram defined by the two vectors.

Anti-commutative

$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

Co-linear operands yield zero vector.

$\mathbf{u} \times \mathbf{u} = \mathbf{0}$ $\quad (k\mathbf{u}) \times \mathbf{u} = \mathbf{0}$.

Cross product with zero vector yields zero vector.

$\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

Not associative

In general: $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ =/= $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

Distributive over vector addition

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

Cross product between coordinate axes

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$ $\quad \mathbf{j} \times \mathbf{k} = \mathbf{i}$ $\quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$

The next chapter discusses rectangular matrices.

Click here to go back to the main menu.

You have reached the End.

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CHAPTER 13 --- Matrix Definition and Simple Operations

In many fields matrices are used to represent objects and operations on those objects. In computer graphics, points and directed line segments are represented with column matrices. Other matrix types represent operations on those objects

Chapter Topics:

- *What a matrix is.*
- *Dimensions of a matrix.*
- *Elements of a matrix.*
- *Matrix addition.*
- *Zero matrix.*
- *Matrix negation.*
- *Matrix subtraction.*
- *Scalar multiplication of a matrix.*
- *The transpose of a matrix*

QUESTION 1:

A column matrix consists of a single column of numbers. Do you think that numbers could be arranged into a several rows and columns?

Click Here after you have answered the question

A good answer might be:

Yes. For a computer science student, the answer is obvious. Numbers are often put into a two-dimensional array.

Definition of Matrix

A matrix is a collection of numbers arranged into a fixed number of rows and columns. Usually the numbers are real numbers. In general, matrices can contain complex numbers but we won't see those here. Here is an example of a matrix with three rows and three columns:

$$\begin{array}{cc} & \text{col 1} \\ \text{row 1} & \begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & 4.6 \\ 4 & -1 & 0 \end{pmatrix} \end{array}$$

The top row is row 1. The leftmost column is column 1. This matrix is a 3x3 matrix because it has three rows and three columns. In describing matrices, the format is:

rows X columns

Each number that makes up a matrix is called an *element* of the matrix. The elements in a matrix have specific locations.

The upper left corner of the matrix is row 1 column 1. In the above matrix the element at row 1 col 1 is the value 1. The element at row 2 column 3 is the value 4.6.

QUESTION 2:

What is the value of the element at row 3 column 1?

Click Here after you have answered the question

What is the value of the element at row 3 column 1?

A good answer might be:

4

Matrix Dimensions

The numbers of rows and columns of a matrix are called its *dimensions*. Here is a matrix with three rows and two columns:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3 \times 2}$$

Sometimes the dimensions are written off to the side of the matrix, as in the above matrix. But this is just a little reminder and not actually part of the matrix. Here is a matrix with different dimensions. It has two rows and three columns. This is a different "data type" than the previous matrix.

$$\begin{pmatrix} -1 & 0 & 1 \\ 5 & 3 & 4 \end{pmatrix}_{2 \times 3}$$

QUESTION 3:

What do you suppose a **square matrix** is? Hint: here is an example:

$$\begin{pmatrix} 5 & 4 & 3 \\ -4 & 0 & 4 \\ 7 & 10 & 3 \end{pmatrix}$$

Click Here after you have answered the question

A good answer might be:

The number of rows == the number of columns

Square Matrix

In a square matrix the number of rows equals the number of columns. In computer graphics, square matrices are used for transformations and are very common.

$$\begin{pmatrix} 5.12 \\ -4.08 \\ 0.0 \\ 1.0 \end{pmatrix}_{4x1}$$

A **column matrix** consists of a single column. It is a N x 1 matrix. These notes, and most computer graphics texts, use column matrices to represent geometrical vectors. At left is a 4 x 1 column matrix. A **row matrix** consists of a single row. (Some books use row matrices to represent geometical vectors).

Some books call a column matrix a *column vector* and call a row matrix a *row vector*. This is OK, but can be ambiguous. In these notes the word *vector* will be used for a geometric object (a directed line segment). In these notes, column matrices will be used to represent vectors and will also be used to represent geometric points.

QUESTION 4:

What are square matrices used for?

Click Here after you have answered the question

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CHAPTER 14 --- Matrix times Column Matrix

A matrix and a column matrix can be multiplied together if their dimensions allow for it. This is a fundamental operation in modern computer graphics (and in many other applictions). It is so fundamental that it is implemented in silicon in all but the most crude of graphics cards. When a 3D graphics program is running the operation is performed millions of times per second.

Chapter Topics:

- *Dimensions of the operands and of the result.*
- *Matrix-column matrix multiplication.*
- *row matrix-Matrix multiplication.*
- *How each element of the result is formed by a dot product.*

You have seen several multiplication-like operations before:

- Scalar times a scalar, resulting in a scalar.
- Scalar times a column matrix, resulting in a column matrix.
- Column matrix dot-product with a column matrix, resulting in a Scalar.

QUESTION 1:

What type of object (do you suppose) is the result of a matrix times a column matrix?

Click Here after you have answered the question

What type of object (do you suppose) is the result of a matrix times a column matrix?

A good answer might be:

A matrix times a column matrix results in a column matrix.

Matrix times Column Matrix

Here is an example of a matrix times a column matrix:

$$\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}_{2x2} \begin{pmatrix} 4 \\ 5 \end{pmatrix}_{2x1} = \begin{pmatrix} 4-10 \\ 0+15 \end{pmatrix}_{2x1} = \begin{pmatrix} -6 \\ +15 \end{pmatrix}_{2x1}$$

Matrix times column matrix

The details of how this is done will be explained later. For now, look at the dimensions of the operands and of the result. A matrix times column matrix product can be formed if the dimensions look like this:

```
Matrix   x   Col. Matrix   =   Resulting Col. Matrix

R x N        N x 1         =         R x 1
    --------
      same
```

For the above example:

```
Matrix   x   Col. Matrix   =   Resulting Col. Matrix

2 x 2        2 x 1         =         2 x 1
    --------
      same
```

Remember that the order of the dimensions is "Row x Column" so that an R x 1 matrix is a column matrix (because it has 1 column).

QUESTION 2:

What are the dimensions of the result, below:

```
Matrix    x   Col. Matrix    =   Resulting Col. Matrix

5 x 5           5 x 1    =        ? x ?
```

Click Here after you have answered the question

A good answer might be:

```
Matrix    x   Col. Matrix    =   Resulting Col. Matrix

5 x 5          5 x 1         =         5 x 1
```

More Practice

For each row in the following table, decide if the matrix times column matrix product can be formed and what the dimensions of the result will be.

Matrix	Col. Matrix	Result
3 x 3	3 x 1	?
3 x 2	2 x 1	?
2 x 3	2 x 1	?
4 x 3	3 x 1	?
3 x 5	5 x 1	?

QUESTION 3:

Do you think that it is possible to form a product that looks like this:

```
Col. Matrix  x  Matrix   =   Result

  2 x 1         2 x 2    =    ???
```

(Hint: do the dimensions look compatible?)

Click Here after you have answered the question

Do you think that it is possible to form a product that looks like this:

```
     Col. Matrix  x  Matrix   =   Result

       2 x 1         2 x 2    =    ???
```

A good answer might be:

No, this is not possible

Column Matrix on the Right, only

When a rectangular matrix and a column matrix are multiplied together, the column matrix will always be on the right of the rectangular matrix. It is only possible to multiply the two if they are *conformant*. Two matrices are conformant if the "inner dimension" of the each matrix is equal:

```
     Matrix      x     Column Matrix   =   Result

     N x M              M x 1          =   N x 1
         ----------------
         inner dimensions
            are equal
```

QUESTION 4:

Do you think that it is possible to form a product that looks like this:

```
     Row Matrix  x  Col. Matrix   =   Scalar

     1 x 2            2 x 1       =   1 x 1
```

[Click Here after you have answered the question](#)

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CHAPTER 15 --- Product of Rectangular Matrices

This chapter discusses the product of two rectangular matrices. This product is an extension of the ideas of the previous chapter. Each element of the product is formed by a dot product.

Chapter Topics:

- *Dimensions of matrices that can be multiplied.*
- *Element of result formed by dot product.*
- *Associative property.*
- *Not commutative.*
- *Distributive over addition.*
- *Product of zero matrix.*

Matrix-matrix products are used in computer graphics to create the transformation matrices that operate on points and vectors. In a "first person game" such as DOOM the landscape is modeled with points and vectors. The constantly changing view you see as the hero moves through the landscape is created by transforming those points and vectors.

QUESTION 1:

Is it possible to multiply the following two matrices? $\mathbf{A}_{R\times N}\ \mathbf{B}_{N\times C}$

Click Here after you have answered the question

Is it possible to multiply the following two matrices? $\mathbf{A}_{R\times N}\ \mathbf{B}_{N\times C}$

A good answer might be:

Yes. The inner dimension "N" matches.

Dimensions of the Result

If two rectangular matrices are put in order so that the inner dimension is the same in each, then the matrices are *conformant* and can be multiplied. The result is (in general) a rectangular matrix:

$$\mathbf{A}_{R\times N}\ \mathbf{B}_{N\times C} = \mathbf{D}_{R\times C}$$

Look at the dimensions in the following product (for now, ignore how the elements were calculated):

$$\begin{pmatrix} 1 & -2 \\ 0 & 3 \\ -1 & 4 \end{pmatrix}_{3x2} \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}_{2x2} = \begin{pmatrix} 4+2 & 1-4 \\ 0-3 & 0+6 \\ -4-4 & -1+8 \end{pmatrix}_{3x2} = \begin{pmatrix} 6 & -3 \\ -3 & 6 \\ -8 & 7 \end{pmatrix}_{3x2}$$

The the product **AB** (if it can be formed) has the same number of rows as **A** and the same number of columns as **B**. You can think of this as "canceling" the inner dimension.

QUESTION 2:

What dimensions will the following product have: $\mathbf{A}_{4\times 4}\ \mathbf{B}_{4\times 2} = \mathbf{C}_{?\ \times\ ?}$

Click Here after you have answered the question

A good answer might be:

$\mathbf{A}_{4\times4}\ \mathbf{B}_{4\times2} = \mathbf{C}_{4\times2}$

Practice with Dimensions

Column matrices and row matrices are a special case of rectangular matrices, so the rule covers them, as well. You may wish to practice with the following. Determine the dimensions of the product of each pair of matrices. Click on the button to check your answer.

Matrix	Matrix	Result	Matrix	Matrix	Result
4 × 3	3 × 2	?	3 × 2	2 × 2	?
2 × 3	2 × 3	?	4 × 3	3 × 1	?
3 × 5	5 × 1	?	3 × 1	1 × 4	?

It always pays to check the dimensions of the result before you do the arithmetic. For example, the last product (in the table) is a bit odd, and it is easy to get mixed up without knowing what the result should look like.

QUESTION 3:

If the product $\mathbf{A}_{n\times m}\ \mathbf{B}_{m\times p}$ can be formed, will it always be possible to form the product $\mathbf{B}_{m\times p}\ \mathbf{A}_{n\times m}$?

Click Here after you have answered the question

If the product $\mathbf{A}_{n\times m}\,\mathbf{B}_{m\times p}$ can be formed, will it always be possible to form the product $\mathbf{B}_{m\times p}\,\mathbf{A}_{n\times m}$?

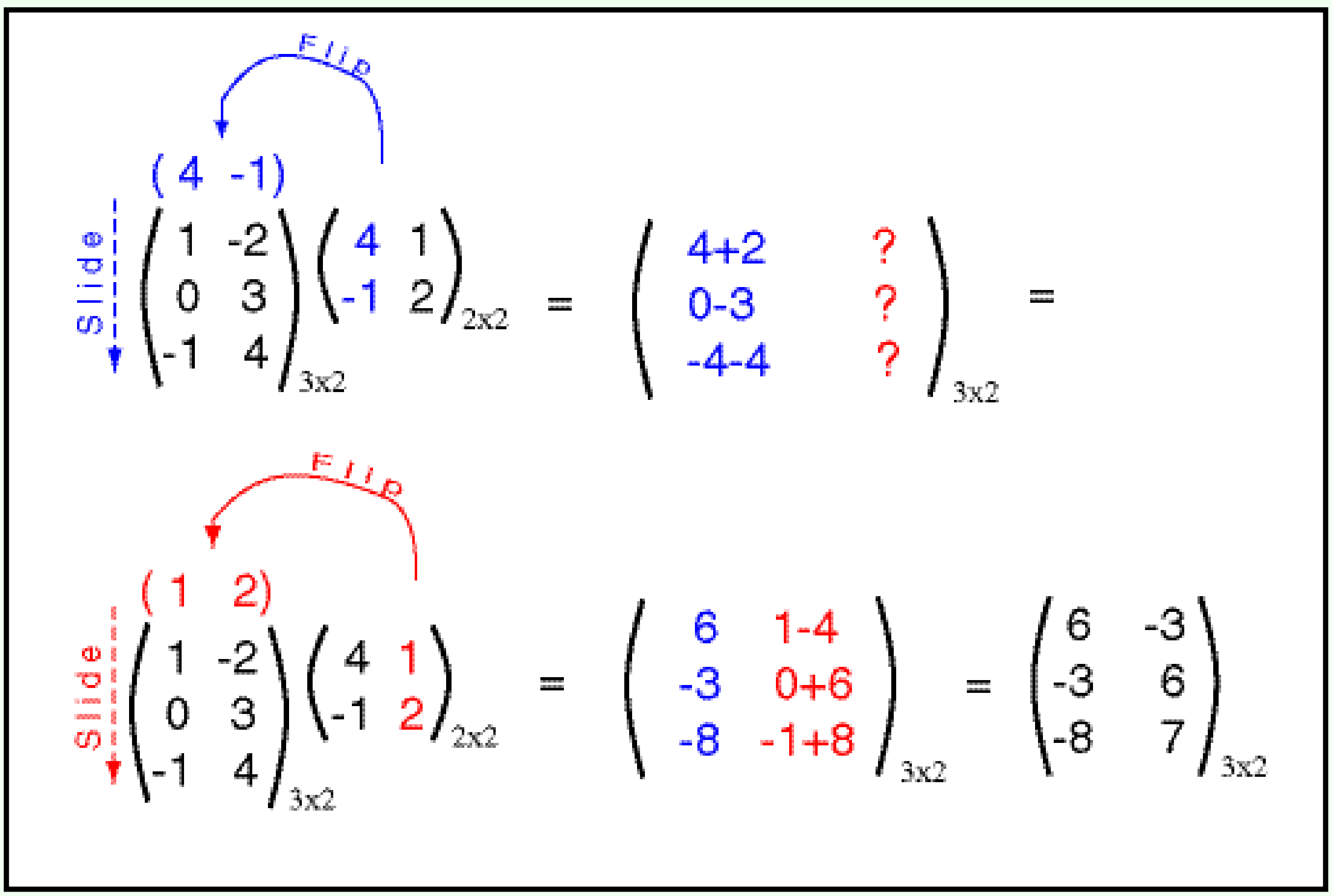
A good answer might be:

No. If n =/= p then **BA** can't be formed. Later you will see that even if both products can be formed, it is rare that **AB** = **BA**.

Forming the Matrix-Matrix Product

The matrix-matrix product **AB** is calculated by forming the dot product of each row of **A** with each column of **B**. Of course, the number of columns in **A** equal to the number of rows in **B**.

As in the previous chapter, flip a column of **B** so that its elements align with the rows of **A**. Form the dot product, which becomes row 1 col 1 of the result. Now slide down one row to form the next dot product, which becomes row 2 col 1 of the result. Continue down until the last row.



When you are finished with first column of **B**, move on to the next column and do the same thing. Continue with each column of **B** until all elements are calculated.

Important: Element ij of the result = dot product of row i of **A** with column j of **B**.

QUESTION 4:

Say that you are forming the product $\mathbf{A}_{5\times3}\ \mathbf{B}_{3\times2} = \mathbf{C}_{5\times2}$

What row and column are used to calculate the 3rd row 2nd column of **C** ?

Click Here after you have answered the question

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CHAPTER 16 --- Identity Matrix and Matrix Inverse

This chapter examines further properties of the matrix-matrix product and two special types of matrices.

Chapter Topics:

- *Identity Matrix.*
- *Matrix Inverse.*
- *Singular and non-singular Matrices.*
- *Determinant of a Singular Matrix.*

For this chapter all matrices will be square.

QUESTION 1:

(Review:) What is a square matrix?

Click Here after you have answered the question

A good answer might be:

A square matrix is one where the number of rows equals the number of columns.

Square Matrices

Computer graphics uses square matrices almost exclusively. Most transformations, such as changing the point of view in a 3D world or projecting it onto a 2D screen involve square matrices.

Square matrices have properties that are not in general shared by other rectangular matrices. To get a taste of one of these properties, perform the following multiplication:

A	**I**	=	**C**
$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	=	$\begin{pmatrix} & \\ & \end{pmatrix}$

Try changing some of the elements of **A** (don't change elements of **I**) and try the product again. Only integer elements will work in this demonstration.

QUESTION 2:

What is true of the product **AI** ?

Click Here after you have answered the question

A good answer might be:

AI = **A**

Pre-multiplication by I

Now perform the following multiplication.

I	A	=	C
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$	=	$\begin{pmatrix} & \\ & \end{pmatrix}$

Try changing some of the elements of **A** (don't change elements of **I**) and try the product again. Only integer elements will work in this demonstration.

QUESTION 3:

What is true of the product **IA** ?

Click Here after you have answered the question

A good answer might be:

IA = **A**

Identity Matrix

There is no unique matrix that works as an identity for matrices of all dimensions. Instead, for each set of square matrices of dimensions N×N there is a matrix $\mathbf{I}_{N\times N}$ that works as an identity.

If $\mathbf{I}_{N\times N}$ is the N-dimensional identity matrix, then its elements are 1 on the main diagonal and 0 elsewhere.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The *main diagonal* consists of those elements $[I]_{m\ m}$ where the row number is equal to the column number. At left is an example of an identity matrix.

The 1s must run down that particular diagonal; it won't work with the other diagonal. For square matrices of dimension n:

$$\mathbf{I}_{n\times n}\ \mathbf{A}_{n\times n} = \mathbf{A}_{n\times n}\ \mathbf{I}_{n\times n} = \mathbf{A}_{n\times n}$$

The identity matrix works like the identity of scalar arithmetic, the value one: 1a = a1 = a.

QUESTION 4:

Is there another N×N matrix that works like **I**?

Click Here after you have answered the question

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A good answer might be:

No, they have the same dimensions, but corresponding elements are not equal.

Matrix Addition

If two matrices have the same number of rows and same number of columns, then the *matrix sum* can be computed:

> If **A** is an MxN matrix, and **B** is also an MxN matrix, then their sum is an MxN matrix formed by adding corresponding elements of **A** and **B**

Here is an example of this:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -1 & 2 \\ 1 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 1 & 7 \\ 4 & 0 \end{pmatrix}$$

Of course, in most practical situations the elements of the matrices are real numbers with decimal fractions, not the small integers often used in examples.

QUESTION 7:

What 3x2 matrix could be added to a second 3x2 matrix without changing that second matrix?

Click Here after you have answered the question

What 3x2 matrix could be added to a second 3x2 matrix without changing that second matrix?

A good answer might be:

The 3x2 matrix that has all its elements zero.

Zero Matrix

A zero matrix is one which has all its elements zero. Here is a 3x3 zero matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3 \times 3} = \mathbf{0}$$

The name of a zero matrix is a bold-face zero: **0**, although sometimes people forget to make it bold face. Here is an interesting problem:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 10 & 7.5 \\ 2 & 9 & -3.2 \\ -2 & 0 & 5 \end{pmatrix} = ?$$

QUESTION 8:

Form the above sum. No electronic calculators allowed!

Click Here after you have answered the question

A good answer might be:

Of course, the sum is the same as the non-zero matrix.

Rules for Matrix Addition

You should be happy with the following rules of matrix addition. In each rule, each matrix has the same dimensions as the others.

> **A** + **B** = **B** + **A**
> **A** + **0** = **0** + **A** = **A**
> **0** + **0** = **0**

These look the same as some rules for addition of real numbers. (**Warning!!** Not all rules for matrix math look the same as for real number math.)

The first rule says that matrix addition is *commutative*. This is because ordinary addition is being done on the corresponding elements of the two matrices, and ordinary (real) addition is commutative:

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \\ 12 & 14 & 16 \end{pmatrix} = \begin{pmatrix} 1+0 & 3+2 & 5+4 \\ 7+6 & 9+8 & 11+10 \\ 13+12 & 15+14 & 17+16 \end{pmatrix}$$

$$= \begin{pmatrix} 0+1 & 2+3 & 4+5 \\ 6+7 & 8+9 & 10+11 \\ 12+13 & 14+15 & 16+17 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \\ 12 & 14 & 16 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{pmatrix}$$

QUESTION 9:

Do you think that: (**A** + **B**) + **C** = **A** + (**B** + **C**)

Click Here after you have answered the question

Do you think that: (**A** + **B**) + **C** = **A** + (**B** + **C**)

A good answer might be:

Yes---this is another rule that works like real number math.

Practice with Matrix Addition

Here is another matrix addition problem. Mentally form the sum (or use a scrap of paper):

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix} = ?$$

Hint: this problem is not as tedious as it might at first seem.

QUESTION 10:

What is the sum?

Click Here after you have answered the question

A good answer might be:

$$\begin{pmatrix}1, & 3\end{pmatrix}\begin{pmatrix}-1\\4\end{pmatrix} = -1*1 + 3*4 = -1 + 12 = 11$$

Flip the Column

In forming these dot products, it is useful to think of "flipping" the column to align it with the row, and then multiplying and adding up corresponding elements:

$$\begin{pmatrix}1, & 3\end{pmatrix}\begin{pmatrix}-1\\4\end{pmatrix} \xrightarrow{\text{flip}} \begin{matrix}(-1,\ 4)\\(1,\ 3)\end{matrix} = -1*1 + 3*4 = 11$$

Here is a second example:

$$\underset{1\text{x}3}{\begin{pmatrix}2, & 0, & 1\end{pmatrix}}\underset{3\text{x}1}{\begin{pmatrix}1\\2\\3\end{pmatrix}} \xrightarrow{\text{flip}} \underset{1\text{x}1}{\begin{matrix}(1,\ 2,\ 3)\\(2,\ 0,\ 1)\end{matrix}} = 2+0+3 = 5$$

QUESTION 6:

Form the following product:

$$\begin{pmatrix} 1, 1, 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Click Here after you have answered the question

A good answer might be:

$$\begin{pmatrix} 1, 1, 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1*1 + 1*0 + 3*-1 = 1 - 3 = -2$$

More Practice

Compute each dot product in the following table. Check your answer in the "Result" column.

First Operand	Second Operand	Result
$\begin{pmatrix} 1, 1, 3 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	?
$\begin{pmatrix} -1, 4, 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	?
$\begin{pmatrix} 0, 2, -2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$	?

Now examine the following matrix times column matrix:

$$\underset{3x3}{\begin{pmatrix} 1, & 1, & 3 \\ -1, & 4, & 1 \\ 0, & 2, & -2 \end{pmatrix}} \underset{3x1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} = \underset{3x1}{\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}}$$

QUESTION 7:

Fill in the result column matrix with a reasonable guess. (Hint: look at the results in the table.)

Click Here after you have answered the question

Do you think that it is possible to form a product that looks like this:

```
Row Matrix  x  Col. Matrix   =   Scalar

1 x 2             2 x 1      =   1 x 1
```

A good answer might be:

Yes. It looks like it can be done.

Dot Product as Matrix Multiplication

Just by looking at the dimensions, it seems that this can be done. Of course, that is not a proof that it can be done, but it is a strong hint. Here is an example:

$$\underset{1\text{x}2}{\begin{pmatrix} 4, & 5 \end{pmatrix}} \underset{2\text{x}1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \underset{1\text{x}1}{4+0} = 4$$

Dot Product

It might look slightly odd to regard a scalar (a real number) as a "1 x 1" object, but doing that keeps things consistent. Notice that this example looks like a dot product. In fact, it is. Knowing that, you know how the product is formed:

$$\underset{1\text{x}2}{\begin{pmatrix} a_0, & a_1 \end{pmatrix}} \underset{2\text{x}1}{\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}} = a_0 \, b_0 + a_1 \, b_1$$

Dot Product

Here is another example, with the result left blank:

$$\underset{1\times 2}{\begin{pmatrix} 1, & 3 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} -1 \\ 4 \end{pmatrix}} = \underline{\qquad ? \qquad}$$

QUESTION 5:

Compute the result.

Click Here after you have answered the question

A good answer might be:

$$\det\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = 1*2 - 3*(-1) = 2+3 = 5$$

Determinant of a Singular Matrix

The determinant of a 2×2 matrix is computed as follows:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a*d - b*c$$

Computing the determinant of larger matrices is more complicated, and rarely done. The determinant is mostly used in discussing matrices, not in computing with them. The following property is often useful:

> The determinant of a singular matrix is zero.

QUESTION 14:

Compute the following determinant (just plug into the above formula):

$$\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

Click Here after you have answered the question

A good answer might be:

$$\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 1*2 - 2*(1) = 2 -2 = 0$$

Rank of a Matrix

The matrix has a zero determinant and is therefor singular. It has no inverse. If you look the matrix you see that it has two identical rows (and two identical columns). In other words, the rows are not independent. If one row is a multiple of another, then they are not independent, and the determinant is zero. (Equivalently: If one col is a multiple of another, then they are not independent, and the determinant is zero.)

The **rank** of a matrix is the maximum number of independent rows (or, the maximum number of independent columns). A square matrix $\mathbf{A}_{n\times n}$ is non-singular only if its rank is equal to n.

QUESTION 15:

What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & 3 & 0 \\ 0 & 0 & 4 & 8 \\ 2 & 4 & 0 & 6 \end{pmatrix}$$

Click Here after you have answered the question

If two matrices contain the same numbers as elements, are the two matrices equal to each other?

A good answer might be:

No, to be equal, two matrices must have the same dimensions, and must have the same values in the same positions.

Matrix Equality

For two matrices to be equal, they must have

1. The same dimensions.
2. Corresponding elements must be equal.

In other words, say that $\mathbf{A}_{\mathbf{n \times m}} = [a_{ij}]$ and that $\mathbf{B}_{\mathbf{p \times q}} = [b_{ij}]$.
Then $\mathbf{A} = \mathbf{B}$ if and only if n=p, m=q, and $a_{ij}=b_{ij}$ for all i and j in range.

Here are two matrices which are not equal even though they have the same elements.

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3 \times 2} =/= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2 \times 3}$$

QUESTION 6:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} =?= \begin{pmatrix} 6 & 4 \\ 5 & 2 \\ 1 & 3 \end{pmatrix}$$

Here is another pair of matrices.

Are these two matrices equal?

Click Here after you have answered the question

Is there another N×N matrix that works like **I**?

A good answer might be:

No. This is not quite obvious, but since 1 is unique you might suspect that **I** is unique.

I is Unique

It is easy to show that this is so. Suppose you had a matrix **Z** and that

(1) **ZA** = **A**

for any **A**. We know how **I** works:

(2) **BI** = **B**

for any **B**. Now use (1) with **I** replacing **A**:

(3) **ZI** = **I**

Now use (2), substituting **Z** for **B**:

(4) **ZI** = **Z**

Looking at (3) and (4):

(4) **Z** = **I**

It is also true that it works with the other order: **AZ** = **A**.

QUESTION 5:

What is the transpose of **I**?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T$$

Click Here after you have answered the question

A good answer might be:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transpose of I

For any dimension: $\mathbf{I}^T = \mathbf{I}$.

$$\mathbf{I}^T = \mathbf{I}$$

Another useful property is the following. Actually, this is really just an application of the definition of **I**.

$$\mathbf{AIB} = \mathbf{AB}$$

Another easy application of the definition:

$\mathbf{II} = \mathbf{I}$, or $\mathbf{I}^2 = \mathbf{I}$

These are all obvious; you should not have to memorize anything here. But it does help to look them over.

QUESTION 6:

Compute the following:

$$\begin{pmatrix} 0.5 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.5 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 \\ 0 & -12 & -8 \\ 4 & 0 & 4 \end{pmatrix} =$$

Hint: you don't actually need to multiply matrices.

[Click Here after you have answered the question]

A good answer might be:

$$\begin{pmatrix} 0.5 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.5 \end{pmatrix}\begin{pmatrix} 2 & 4 & 6 \\ 0 & -12 & -8 \\ 4 & 0 & 4 \end{pmatrix} = (0.5)\mathbf{I}\begin{pmatrix} 2 & 4 & 6 \\ 0 & -12 & -8 \\ 4 & 0 & 4 \end{pmatrix} =$$

$$0.5\begin{pmatrix} 2 & 4 & 6 \\ 0 & -12 & -8 \\ 4 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -6 & -4 \\ 2 & 0 & 2 \end{pmatrix}$$

Diagonal Matrix

The diagonal matrix (a**I**) is sometimes useful in computer graphics. Say for example that what you want to compute is:

a **ABx**

where a is a scalar and **x** is a column vector. Sometimes it is convenient to think of this as:

(a**I**)**ABx**

Now every operation is matrix multiplication. (This sometimes has a practical advantage with graphcis hardware).

QUESTION 7:

What is (2**I**)(0.5**I**)?

Click Here after you have answered the question

A good answer might be:

$\mathbf{p} = (1\ 2)^T$

Matrix Inverse

It was tedious to figure that out (and would be much worse if **A** were, say, 5 × 5). It would be nice to have a better way. Say that

(1) $\mathbf{q} = \mathbf{A}\mathbf{p}$

for column vectors **p** and **q** and n×n matrix **A** . Is it possible to find a matrix $\mathbf{B}_{n\times n}$ so that

(2) $\mathbf{p} = \mathbf{B}\mathbf{q}$

If it is, then using (1) and (2)

(3) $\mathbf{q} = \mathbf{A}(\mathbf{B}\mathbf{q})$

(4) $\mathbf{q} = (\mathbf{A}\mathbf{B})\mathbf{q}$

If (4) is true, then $(\mathbf{AB}) = \mathbf{I}$. (Remember that **I** is unique). **B**, if it exists, is the **inverse** of **A**, written $\mathbf{A}^{-1}$, and the following is true:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

QUESTION 10:

For any square matrix **A**, is there always going to be an inverse $\mathbf{A}^{-1}$? Hint: consider the zero matrix.

Click Here after you have answered the question

A good answer might be:

No. For example there is no matrix $\mathbf{0}^{-1}$ such that $\mathbf{0}\mathbf{0}^{-1} = \mathbf{I}$

Non-singular

In fact, it is worse than that. There are very many n × n matrices that do not have an inverse. A matrix that does have an inverse is called **non-singular**. A matrix that does not is called singular. If the matrix **A** is non-singular, then:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Here is a matrix that has an inverse:

A	A^-1	=	I

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Rarely is an inverse as easy to find as the above one. Now, say that

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} p = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

QUESTION 11:

What is **p**? Use $\mathbf{A}^{-1}$ from above.

Click Here after you have answered the question

A good answer might be:

$(\mathbf{AB})\ (\mathbf{B}^{-1}\ \mathbf{A}^{-1}) = \mathbf{A}\ (\mathbf{BB}^{-1})\ \mathbf{A}^{-1}$, by associativity.

$= \mathbf{A}\ \mathbf{I}\ \mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$

Properties of the Matrix Inverse

The answer to the question shows that:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\ \mathbf{A}^{-1}$$

Notice that the order of the matrices has been reversed on the right of the "=" . Another sometimes useful property is:

$$(\mathbf{A}^{-1})^{T} = (\mathbf{A}^{T})^{-1}$$

QUESTION 13:

What is the determinant of:

$$\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$$

(If you have forgotten about determinants, or wish you had, don't worry. They won't be used much.)

Click Here after you have answered the question

A good answer might be:

$$\begin{pmatrix} 10 & 0 & 20 \\ 20 & 40 & 10 \\ 0 & 20 & 10 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 1 & 1 \\ 2 & -2 \end{pmatrix} = 10\begin{pmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 1 & 1 \\ 2 & -2 \end{pmatrix} = 10\begin{pmatrix} 2 & -2 \\ 2 & 6 \\ 4 & 0 \end{pmatrix} = 20\begin{pmatrix} 1 & -1 \\ 1 & 3 \\ 2 & 0 \end{pmatrix}$$

Associative

Multiply the first two following matrices together. Click on the = to check your result. Then multiply the result with the third matrix.

Problem		Partial Result		Result
$\left(\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}\right)\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$	=	$\begin{pmatrix} ?\,? \\ ?\,? \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$	=	$\begin{pmatrix} ?\,? \\ ?\,? \end{pmatrix}$

Now do the problem again, but this time start by multiplying the last two matrices.

Problem		Partial Result		Result
$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\left(\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\right)$	=	$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} ?\,? \\ ?\,? \end{pmatrix}$	=	$\begin{pmatrix} ?\,? \\ ?\,? \end{pmatrix}$

The final answer is the same for both ways of doing the problem. This demonstates the fact that matrix multiplication is **associative**:

(AB)C = **A(BC)**

Of course, the inner dimension of **A** and **B** must be the same, and the inner dimension of **B** and **C** must be the same. Usually a product of three matrices is written **ABC**.

QUESTION 11:

Say that $\mathbf{A}_{5\times5}\ \mathbf{B}\ \mathbf{C}_{3\times4} = \mathbf{D}$.

What are the dimensions of **B** and **D**?

Click Here after you have answered the question

A good answer might be:

$\mathbf{A}_{5\times 5}\ \mathbf{B}_{5\times 3}\ \mathbf{C}_{3\times 4} = \mathbf{D}_{5\times 4}$.

Distributive Property

The distributive property deals with a matrix expression that contains both matrix multiplication and matrix addition. Go through the following steps to demonstrate the property.

1. Add the right two matrices together, click = to check your result, below.

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix} \right) \ \boxed{=}$$

2. Multiply the first matrix times the result. Click "=" to check your result.

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} \ \boxed{=} \ \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

3. Now, work the problem again by forming the product of the left matrix and each of the others.

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix} \boxed{=}$$

4. Finally, add up the two products.

$$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} + \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} \boxed{=} \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Both results are the same, demonstrating that matrix multiplication is distributive over matrix addition:

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB}+\mathbf{AC}$$

It is also true that

$$(\mathbf{X} + \mathbf{Y})\mathbf{Z} = \mathbf{XZ}+\mathbf{YZ}$$

For more practice, change the elements in the matrices of step 1 (only), and then work the problem again. When you make changes, replace the contents of a cell with an integer (only) and click outside of the cell to make the change.

QUESTION 12:

Calculate the following:

$$\begin{pmatrix} \cos w & 1 \\ 0 & \sin w \end{pmatrix} \begin{pmatrix} x & 2 \\ 1 & y \end{pmatrix}$$

Click Here after you have answered the question

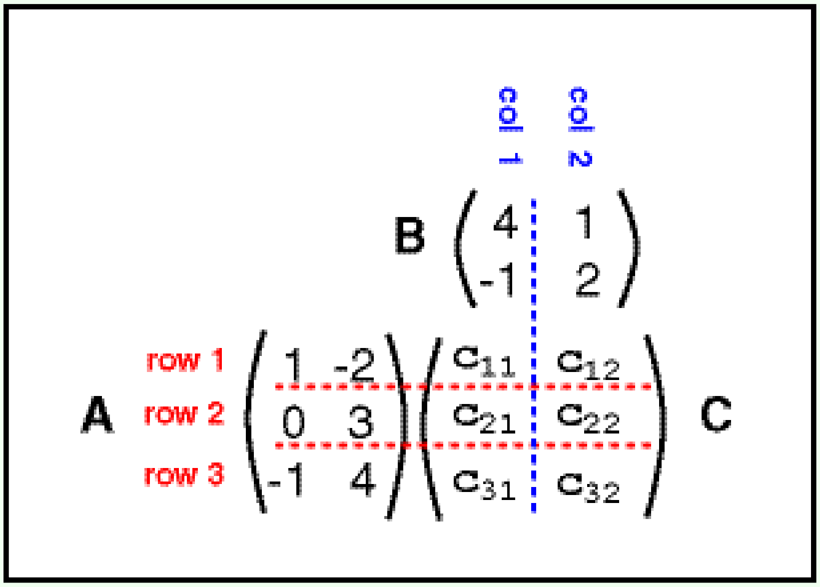
Say that you are forming the product $\mathbf{A}_{5\times3}\,\mathbf{B}_{3\times2} = \mathbf{C}_{5\times2}$

What row and column are used to calculate the 3rd row 2nd column of **C** ?

A good answer might be:

The 3rd row of **A** and the 2nd column of **B**

Another Way to Visualize



If you are also following a text book don't be surprised if it shows a different way to do matrix-matrix multiplication. There are several ways to think about it, although the results will be the same.

The diagram shows another way to visualize the product **AB = C**. Write **A** to the left and **B** above, then draw horizontal lines and vertical lines to divide **C** into cells. This will automatically form the correct number of cells.

The number to put in each cell is the dot product of the corresponding row of **A** and column of **B** .

QUESTION 5:

What dot product is done for:

- c_{11} ?
- c_{32} ?

Click Here after you have answered the question

A good answer might be:

$$\begin{pmatrix} \cos(w) & 1 \\ 0 & \sin(w) \end{pmatrix}\begin{pmatrix} x & 2 \\ 1 & y \end{pmatrix} = \begin{pmatrix} x\cos(w) + 1 & 2\cos(w) + \\ \sin(w) & y\sin(w) \end{pmatrix}$$

Although most of our examples have used integers, don't forget that matrix elements can be real numbers or variables.

Summary of Rules (so far)

Here is a list of rules that this chapter has discussed. Each rule assumes that the matrices can be multiplied (their dimensions match correctly). Click on a rule to review the page that discusses it.

Conformant Matrices:

if $\mathbf{A}_{M\times K}\,\mathbf{B}_{K\times N} = \mathbf{C}$, then $\mathbf{C} = \mathbf{C}_{M\times N}$

Not Commutative:

AB =/= **BA**, except rarely.

Associative:

A(BC) = **(AB)C** = **ABC**

Scalar factor:

a(**AB**) = (a**A**)**B** = a**AB** = **A**(a**B**)

Distributive:

A(**B** + **C**) = **AB** + **AC**

Distributive:

(**A** + **B**)**C** = **AC** + **BC**

Multiplication by zero:

0A = **0**,   for the zero matrix   **0**

You have reached the end of this chapter. The next chapter will discuss further properties of matrix-matrix multiplication.

Click here to go back to the main menu.

You have reached the End.

Is it always true that **AB** = **BA**?

A good answer might be:

NO! Even when both products can be formed they are rarely equal.

Not Commutative

For example, calculate both of the following products (you might wish to use paper and pencil):

Problem	Think First	Result
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$	Answer	$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$
$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	Answer	$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$

The results show that in general for matrices **A** and **B**, **AB** =/= **BA**.

QUESTION 8:

Is $\mathbf{A}_{4\times 4}\ \mathbf{0}_{4\times 4} == \mathbf{0}_{4\times 4}\ \mathbf{A}_{4\times 4}$, where **0** is the zero matrtix?

Click Here after you have answered the question

A good answer might be:

Yes, $\mathbf{A}_{4\times4}\, \mathbf{0}_{4\times4} == \mathbf{0}_{4\times4}\, \mathbf{A}_{4\times4} == \mathbf{0}_{4\times4}$

For some matrices **AB** == **BA**. But not in general.

More Practice

It would be good at this point to practice matrix multipliction. The goal is to interalize the idea so that it becomes one of your active concepts, not a faintly recalled notion you have to look up each time.

In the following, mentally (or on paper) multiply the two matrices. Click on the "=" button to check your answer. Change elements in **A** and **B** and work the problem again. To change an element, replace the contents of a cell with an integer (only) and then click outside of the cell.

A	B	=	C
$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$	=	$\begin{pmatrix} & \\ & \end{pmatrix}$

Try some interesting problems, such as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} =$$

QUESTION 9:

Do you suspect that x(**AB**) == (x**A**)**B** for scalar (real number) x?

Click Here after you have answered the question

A good answer might be:

YES   x(**AB**) == (x**A**)**B** for scalar x.

Scalar Factor

Recall that x**A** means that every element of **A** is multiplied by x. Here is a demonstration of the rule:

$$\left[\alpha\begin{pmatrix}1&2\\3&4\end{pmatrix}\right]\begin{pmatrix}5&6\\7&8\end{pmatrix} = \begin{pmatrix}\alpha1&\alpha2\\\alpha3&\alpha4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix} = \begin{pmatrix}\alpha1\cdot5+\alpha2\cdot7 & \alpha1\cdot6+\alpha2\cdot8\\ \alpha3\cdot5+\alpha4\cdot7 & \alpha3\cdot6+\alpha4\cdot8\end{pmatrix}$$

$$\alpha\left[\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}5&6\\7&8\end{pmatrix}\right] = \alpha\begin{pmatrix}1\cdot5+2\cdot7 & 1\cdot6+2\cdot8\\ 3\cdot5+4\cdot7 & 3\cdot6+4\cdot8\end{pmatrix} =$$

Often a matrix-matrix multiplication can be simpified by factoring one of the matrices. See the question.

QUESTION 10:

Compute the following:

$$\begin{pmatrix}10&0&20\\20&40&10\\0&20&10\end{pmatrix}\begin{pmatrix}-2&2\\1&1\\2&-2\end{pmatrix}$$

Click Here after you have answered the question

A good answer might be:

Yes, the dimensions are compatible:

$$\underset{1\times 3}{\begin{pmatrix} 1, & 2, & -1 \end{pmatrix}} \underset{3\times 2}{\begin{pmatrix} 1, & 0 \\ -1, & 4 \\ 0, & 3 \end{pmatrix}} = \underset{1\times 2}{\begin{pmatrix} 1*1 - 1*2 - 1*0, & 0*1 + 2*4 - 1*3 \end{pmatrix}}$$

$$= \begin{pmatrix} -1, & 5 \end{pmatrix}$$

Row Matrix times Matrix as Several Dot Products

From the answer, you can see that row matrix times matrix is done by forming several dot products. However, here it is the *columns of the matrix* that get flipped:

flip 1

$$\overset{(1,\ -1,\ 0)}{\underset{1\times 3}{\begin{pmatrix}1, & 2, & -1\end{pmatrix}}} \underset{3\times 2}{\begin{pmatrix}1, & 0\\ -1, & 4\\ 0, & 3\end{pmatrix}} = \underset{1\times 2}{\begin{pmatrix}1*1 - 1*2 -1*0, & \_\_\_\_\_\_\end{pmatrix}}$$

flip 2

$$= \overset{(0,\ 4,\ 3)}{\begin{pmatrix}1, & 2, & -1\end{pmatrix}} \begin{pmatrix}1, & 0\\ -1, & 4\\ 0, & 3\end{pmatrix} = \begin{pmatrix}-1, & 0*1 + 2*4 - 1*3\end{pmatrix}$$

$$= \begin{pmatrix}-1, & 5\end{pmatrix}$$

QUESTION 10:

Is this type of product possible:

```
Matrix  x  Row Matrix   =   ?

3 x 4        1 x  3     =   ?
```

[Click Here after you have answered the question]()

A good answer might be:

No. The dimensions are not conformant.

Row Matrix always on the Left

When a row matrix is multiplied by a rectangular matrix, the row matrix must be on the left. Look at the dimensions of the operands and of the result:

```
     Row Matrix  x  Matrix   =   Result Row Matrix

       1 x M        M x C           1 x C
```

A 1xC matrix is a row matrix of "C" number of columns. For each row in the following table, decide if the product can be formed and what the dimensions of the result will be.

First Operand	Second Operand	Result
3 x 3	3 x 1	?
1 x 2	2 x 2	?
1 x 2	2 x 3	?
4 x 4	3 x 1	?
1 x 5	5 x 3	?

QUESTION 11:

Perform the following multiplication:

$$\underbrace{\begin{pmatrix} 4, & 5 \end{pmatrix}}_{1x2} \underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}}_{2x2} =$$

Click Here after you have answered the question

A good answer might be:

$\mathbf{A}^{-1}$	$\mathbf{A}$	$\mathbf{p}$	=	$\mathbf{A}^{-1}$	$\mathbf{q}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{p} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\mathbf{I}\ \mathbf{p} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\mathbf{p} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Unique $\mathbf{A}^{-1}$

This is (hopefully) the same answer you got for **p** by trial and error a few pages ago. If **A** is non-singular (has an inverse) and **Ap** = **q**, then **p** = $\mathbf{A}^{-1}\mathbf{q}$.

The inverse of a non-singular square matrix is unique. One way to see this is that there is only one column matrix **p** that is the solution to **Ap** = **q**, so there must be only one $\mathbf{A}^{-1}$.

It might look like computing $\mathbf{A}^{-1}$ is a useful thing to do. In fact, $\mathbf{A}^{-1}$ is more useful in discussions about matrices and transformations than it is in actual practice. Almost never do you really want to compute a matrix inverse.

For example, say that a column matrix **p** represents a point in a computer graphic world. The viewpoint changes, and the column matrix is transformed to **q** = **Ap**. If you want to talk about reversing the transformation, you talk about $\mathbf{A}^{-1}\mathbf{q}$. But almost always there is an easier way to reverse the transformation than to compute the inverse.

QUESTION 12:

What is $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$?

Click Here after you have answered the question

A good answer might be:

Probably you came up with the correct answer:

$$\underset{3x3}{\begin{pmatrix} 1, & 1, & 3 \\ -1, & 4, & 1 \\ 0, & 2, & -2 \end{pmatrix}} \underset{3x1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} = \underset{3x1}{\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}}$$

Repeated Dot Products

The Matrix-column matrix product is done by forming the dot product of the column matrix with each row of the matrix. Of course, the dimensions of each operand have to allow for this. It helps to flip the column matrix so that its elements align with the elements of the matrix:

$$\underset{3x3}{\begin{pmatrix} 1, & 1, & 3 \\ -1, & 4, & 1 \\ 0, & 2, & -2 \end{pmatrix}} \underset{3x1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \xrightarrow{\text{flip}} \begin{array}{c} \begin{pmatrix} 1, & 0, & -1 \end{pmatrix} \\ \begin{pmatrix} 1, & 1, & 3 \\ -1, & 4, & 1 \\ 0, & 2, & -2 \end{pmatrix} \end{array} = \underset{3x1}{\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}}$$

Here is another example, with the result not yet quite complete:

$$\underset{3x3}{\begin{pmatrix} 0, & 1, & -1 \\ 2, & 1, & 0 \\ 1, & 0, & 2 \end{pmatrix}} \underset{3x1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = \underset{3x1}{\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}}$$

QUESTION 8:

Mentally flip (transpose) the column matrix and compute the result.

Click Here after you have answered the question

A good answer might be:

$$\underbrace{\begin{pmatrix} 0, & 1, & -1 \\ 2, & 1, & 0 \\ 1, & 0, & 2 \end{pmatrix}}_{3x3} \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{3x1} = \underbrace{\begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}}_{3x1}$$

More Flipping Practice

Here are some more products that you may wish to practice with. Remember to (mentally) flip the column matrix and place it ontop of the matrix. Then slide it one row at a time as you compute the dot product.

Problem	Think First	Result
$\begin{pmatrix} 0, & 1, & -1 \\ 2, & 1, & 0 \\ 1, & 0, & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	Answer	$\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$
$\begin{pmatrix} 0, & 1, & -1 \\ 2, & 1, & 0 \\ 1, & 0, & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$	Answer	$\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$
$\begin{pmatrix} 1, & 2 \\ 3, & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	Answer	$\begin{pmatrix} ? \\ ? \end{pmatrix}$

QUESTION 9:

Can the following operation be performed?

$$\begin{pmatrix} 1, & 2, & -1 \end{pmatrix} \begin{pmatrix} 1, & 0 \\ -1, & 4 \\ 0, & 3 \end{pmatrix}$$

(If you want, you can try to do it.)

Click Here after you have answered the question

Is **Ax** equal to **xA** ?

A good answer might be:

No. If one product can be formed, then often the second one has incompatible dimensions. Sometimes both products can be formed, but even then the numbers will usually be different.

More Practice

You might enjoy testing your skills on the following:

Problem	Think First	Result
$\underset{1\times2}{\begin{pmatrix}1, & -2\end{pmatrix}} \underset{2\times2}{\begin{pmatrix}1 & -2 \\ 0 & 3\end{pmatrix}} =$	Answer	$\begin{pmatrix}?, & ?\end{pmatrix}$
$\underset{1\times3}{\begin{pmatrix}3, & 0, & -2\end{pmatrix}} \underset{3\times3}{\begin{pmatrix}0 & -2 & 1 \\ 1 & 0 & 1 \\ -2 & 0 & 1\end{pmatrix}} =$	Answer	$\begin{pmatrix}?, & ?, & ?\end{pmatrix}$
$\underset{1\times3}{\begin{pmatrix}3, & 0, & -2\end{pmatrix}} \underset{3\times3}{\begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}} =$	Answer	$\begin{pmatrix}?, & ?, & ?\end{pmatrix}$

QUESTION 14:

Had enough?

Click Here after you have answered the question

A good answer might be:

Yes.

End of the Chapter

You are done with this chapter. You may wish to flip through the following. Click on a blue subject that interests you to flip back to where it was discussed. To get back here, click on the "back arrow" button of your browser.

- Dimensions of operands and result in matrix times column matrix multiplication.
- Dimensions of operands and result in row matrix times matrix multiplication.
- Conformant matrices.
- The dot product as a matrix product.
- Matrix product formed by many dot products
- Flipping the column matrix in matrix times column matrix multiplication.
- Flipping the matrix columns in row matrix times matrix multiplication.
- Mathematical definitions of these products.
- Steps in performing these products.
- Non-commutative property of matrix products.

The next chapter will discuss Matrix-matrix multiplication.

Click here to go back to the main menu.

You have reached the End.

A good answer might be:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{pmatrix}_{4x4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}_{4x1} = \begin{pmatrix} -1 \\ -1 \\ 3 \\ 2 \end{pmatrix}_{4x1}$$

Steps in Multiplication

By the definitions on the previous page, multiplication always involves dot products of *rows* on the left with *columns* on the right. So, To do any of these calculations:

1. Examine the dimensions of the operands.
   - The inner dimensions must be equal, eg., 1xN and NxC
   - "Cancel" the inner dimensions to find the dimensions of the result, eg., 1xC.
   - Write an empty result with the correct dimensions.
2. Now fill in each element of the result.
   - Each column on the right is "flipped" onto a row on the left.
   - Each element of the result is a dot product.

QUESTION 13:

Is **Ax** equal to **xA** ?

Click Here after you have answered the question

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & 3 & 0 \\ 0 & 0 & 4 & 8 \\ 2 & 4 & 0 & 6 \end{pmatrix}$$

A good answer might be:

Three. The last row is a multiple of the first.

End of the Chapter

You have reached the end this chapter. Here is a list of properties discussed in this chapter.

- **IA** = **AI** = **A**
- **I** is unique.
- $\mathbf{I}^T = \mathbf{I}$
- **AIB** = **AB** **II = I**
- $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- Singular and non-singular matrices.
- Uniqueness of matrix inverse.
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$
- Determinant of a singular matrix.

Click here to go back to the main menu.

You have reached the End.

A good answer might be:

It seems like subtraction could be defined as adding a negation of a matrix.

Matrix Subtraction

If **A** and **B** have the same number of rows and columns, then **A** - **B** is defined as **A** + **(-B)**. Usually you think of this as:

> To form **A** - **B**, from each element of **A** subtract the corresponding element of **B**.

Here is a partly finished example:

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 0 \\ ? & -5 & -2 \\ 0 & 2 & ? \end{pmatrix}$$

Notice in particular the elements in the first row of the answer. The way the result was calculated for the elements in row 1 column 2 is sometimes confusion.

QUESTION 13:

Mentally fill in the two question marks.

Click Here after you have answered the question

A good answer might be:

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 6 & 0 \\ 0 & -5 & -2 \\ 0 & 2 & -6 \end{pmatrix}$$

Transpose

The *transpose* of a matrix is a new matrix who's rows are the columns of the original (which makes its columns the rows of the original). Here is a matrix and its transpose:

$$\begin{pmatrix} 5 & 4 & 3 \\ 4 & 0 & 4 \\ 7 & 10 & 3 \end{pmatrix}^T = \begin{pmatrix} 5 & 4 & 7 \\ 4 & 0 & 10 \\ 3 & 4 & 3 \end{pmatrix}$$

The superscript "T" means "transpose". Another way to look at the transpose is that the element at row r column c if the original is placed at row c column r of the transpose. We will usually work with square matrices, and it is usually square matrices that will be transposed. However, non-square matrices can be transposed, as well:

$$\begin{pmatrix} 5 & 4 \\ 4 & 0 \\ 7 & 10 \\ -1 & 8 \end{pmatrix}^T_{4 \times 2} = \begin{pmatrix} 5 & 4 & 7 & -1 \\ 4 & 0 & 10 & 8 \end{pmatrix}_{2 \times 4}$$

QUESTION 14:

What is the transpose of:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2 \times 3}^{T} = \quad ?$$

Click Here after you have answered the question

A good answer might be:

$$\underset{1\times2}{\begin{pmatrix} 4, & 5 \end{pmatrix}} \underset{2\times2}{\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}} = \underset{1\times2}{\begin{pmatrix} 4+0, & -8+15 \end{pmatrix}} = \underset{1\times2}{\begin{pmatrix} 4, & +7 \end{pmatrix}}$$

Careful Definitions

You may be uneasy with mathematical operations being defined in this casual fashion. Here are some more accurate definitions:

> **Matrix times Column Matrix:** Let **A** be a M x C matrix, and **x** be a C x 1 column matrix. Then the product **Ax** is the M x 1 column matrix whose *i-th* element is the dot product of row i of **A** with **x**.
>
> **Row Matrix times Matrix:** Let **x** be a 1 x R row matrix and **A** be a R x M matrix, Then the product **xA** is the 1 x M row matrix whose *i-th* element is the dot product of **x** with column i of **A**.

QUESTION 12:

Perform the following multiplication:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{pmatrix}_{4\times4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}_{4\times1} =$$

Click Here after you have answered the question

A good answer might be:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^{T}_{2 \times 3} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3 \times 2}$$

A Rule for Transpose

If a transposed matrix is itself transposed, you get the original back:

$$\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^{T}\right)^{T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}^{T} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

This illustrates the rule $(\mathbf{A}^{T})^{T} = \mathbf{A}$.

QUESTION 15:

What is the transpose of:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^{T} = \ ?$$

Click Here after you have answered the question

A good answer might be:

(2**I**)(0.5**I**) = **I**

Product Yields I

It is clear that (2**I**)(0.5**I**) = 2 (0.5)**I** = 1**I** = **I**. But, just for practice, mentally multiply out these matrices the long way:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

The rows from the first matrix "match" the columns of the second so that the dot product is 1 only for the major diagonal. All other dot products are 0. The resulting matrix is **I**.

QUESTION 8:

Say that ax = b for real numbers a and b and real variable x. Solve for x.

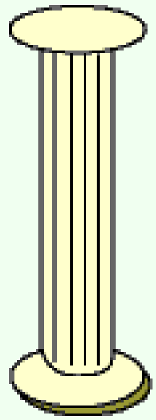
Click Here after you have answered the question

A good answer might be:

Square matrices are used (in computer graphics) to represent geometric transformations.

Names for Matrices

If you sometimes forget whether rows or columns come first, just remember that matrices are build out of rowman columns. A matrix can be given a name. In printed text, the name for a matrix is usually a capital letter in bold face, like **A** or **M**. Sometimes as a reminder the dimensions are written to the right of the letter, as in $\mathbf{B_{3x3}}$.



The elements of a matrix also have names, usually a lowercase letter the same as the matrix name, with the position of the element written as a subscript. So, for example, the 3x3 matrix **A** might be written as:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Sometimes you write $\mathbf{A} = [a_{ij}]$ to say that the elements of matrix **A** are named a_{ij}.

QUESTION 5:

(Thought Question:) If two matrices contain the same numbers as elements, are the two matrices equal to each other?

Click Here after you have answered the question

A good answer might be:

Each element in the result is the negative of the original, as seen below.

Negative of a Matrix

The negation of a matrix is formed by negating each element of the matrix:

$$\mathbf{-A} = -1 \cdot \mathbf{A}$$

So, for example:

$$-1\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{pmatrix}$$

It will not surprise you that **A** + **(-A)** = **0**

QUESTION 12:

Look at the above fact. Can you think of a way to define *matrix subtraction*?

Click Here after you have answered the question

A good answer might be:

Each element of the 3x3 result is 10.

Multipliation of a Matrix by a Scalar

A matrix can be multiplied by a scalar (by a real number) as follows:

> To multiply a matrix by a scalar, multiply each element of the matrix by the scalar.

Here is an example of this. (In this example, the variable *a* is a scalar.)

$$a \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1a & 2a & 3a \\ 4a & 5a & 6a \\ 7a & 8a & 9a \end{pmatrix}$$

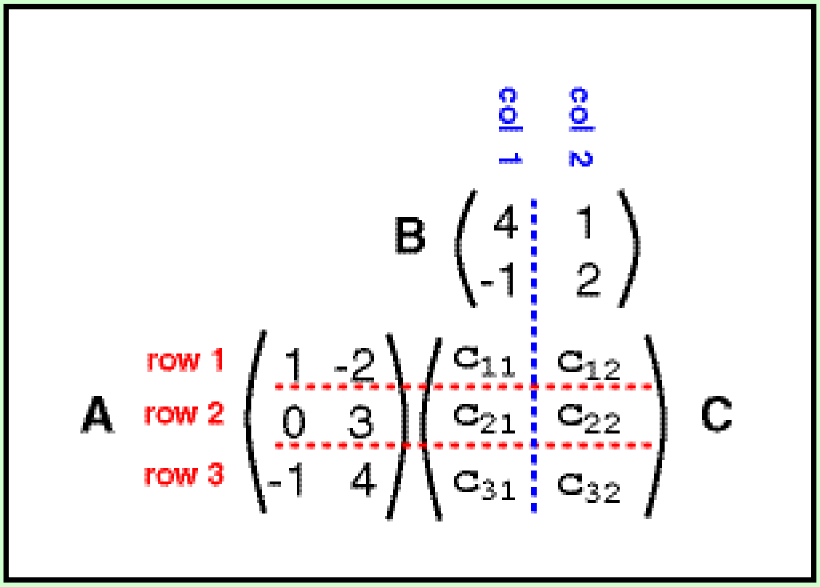
QUESTION 11:

Show the result if the scalar *a* in the above is the value -1.

Click Here after you have answered the question

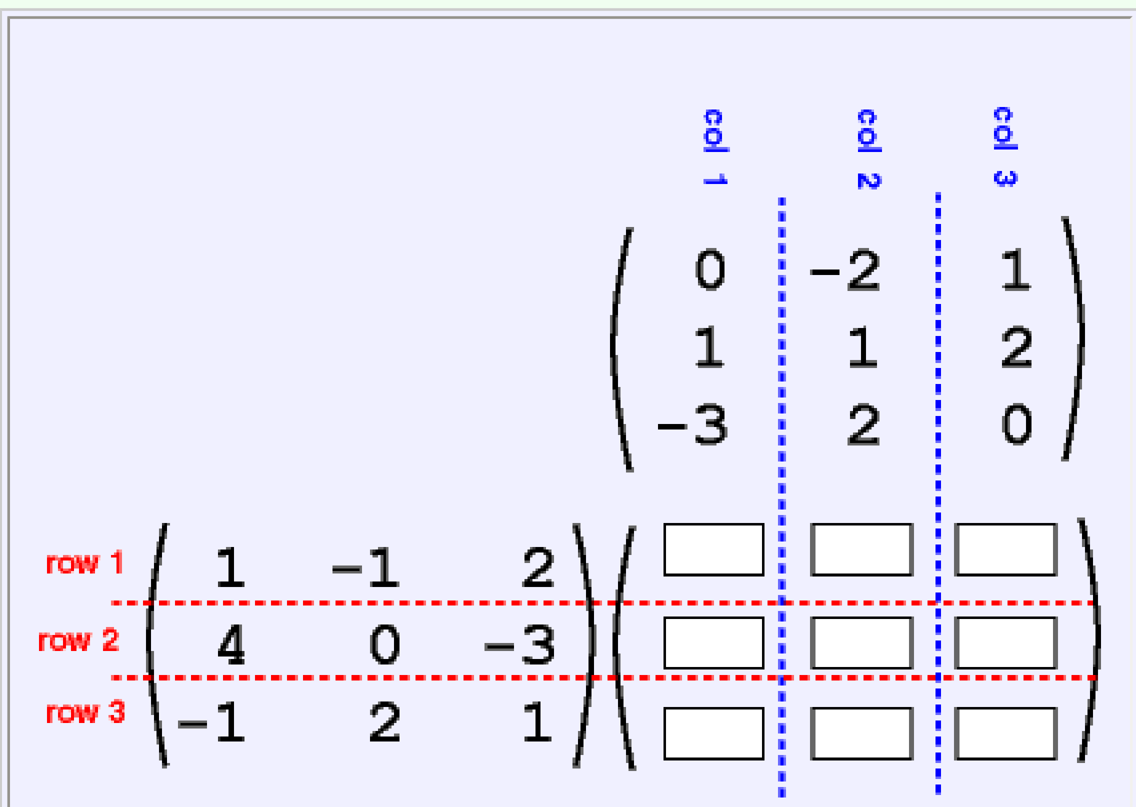
A good answer might be:

- $c_{11} = 1\times4 + -2\times-1 = 6$
- $c_{32} = -1\times1 + 4\times2 = 7$



Multiplication Practice

You are surely eager to perform a matrix-matrix product! In the following diagram, mentally calculate the dot product that belongs in each cell. Then move the mouse over it to see the correct value.



In order for the above to work you must use a recent web browser (such as Netscape Navigator 4.0) and have JavaScript enabled. If it doesn't work, work out the answer with paper and pencil.

QUESTION 6:

Compute the matrix-matrix product.

[Click Here after you have answered the question]

A good answer might be:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

The transpose of a row matrix is a column matrix. And the transpose of a column matrix is a row matrix.

Rule Summary

Here are some rules that cover what has been discussed. You should check that they seem reasonable, rather than memorize them. For each rule the matrices have the same number of rows and columns.

$\mathbf{A} + \mathbf{0} = \mathbf{A}$	$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	$\mathbf{0} + \mathbf{0} = \mathbf{0}$
$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$	$(ab)\mathbf{A} = a(b\mathbf{A})$	$a(\mathbf{A} + \mathbf{B}) = a\mathbf{B} + a\mathbf{A}$
$a\mathbf{0} = \mathbf{0}$	$(-1)\mathbf{A} = \mathbf{-A}$	$\mathbf{A} - \mathbf{A} = \mathbf{0}$
$(\mathbf{A}^T)^T = \mathbf{A}$	$\mathbf{0}^T = \mathbf{0}$	

In the above, a and b are scalars (real numbers). **A** and **B** are matrices, and **0** is the zero matrix of appropriate dimension.

QUESTION 16:

If **A** = **B** and **B** = **C**, then does **A** = **C**?

Click Here after you have answered the question

A good answer might be:

If ax = b, then $(a^{-1})ax = (a^{-1})b$, or $x = (a^{-1})b$ (except when a is zero).

Solve for p

The value (a^{-1}) is the inverse of a, and all non-zero real numbers have an inverse. It would be nice to have a similar idea for square matrices.

Say that

Ap = **q**

for column vectors **p** and **q** and n×n matrix **A** . In computer graphics terms, the point represented by the column matrix **p** has been transformed into the point represented by **q**.

If all know are **A** and **q**, is it possible to figure out what **p** is? For example, say that you know that

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} p = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Try to figure out **p** from the following. Keep changing the elements of **p** until the result is **q**.

A	p	=	q

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \\ \end{pmatrix}$$

QUESTION 9:

Did you figure it out?

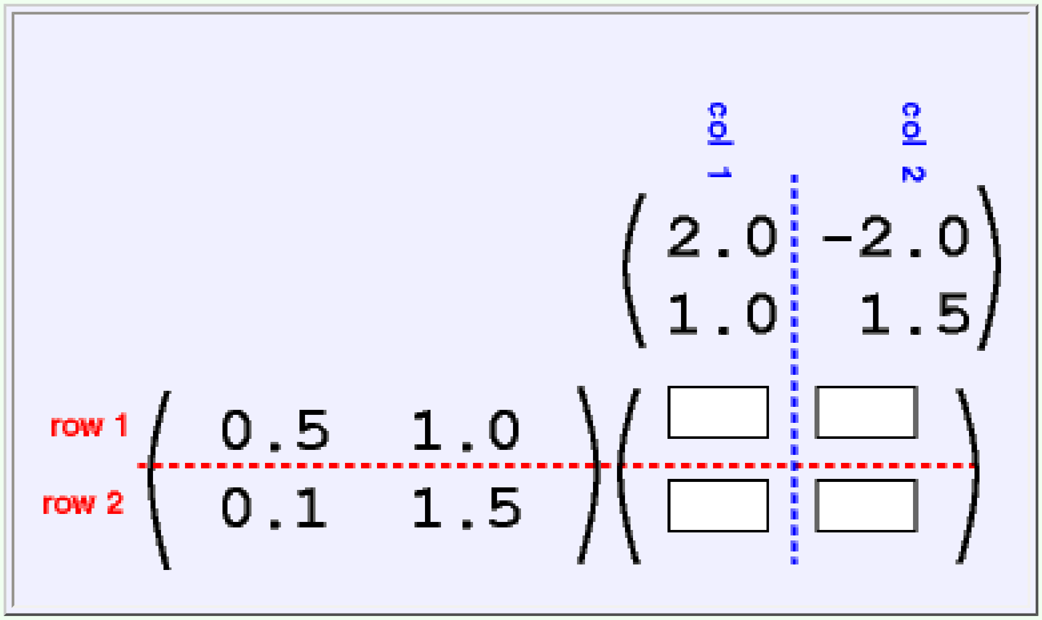
Click Here after you have answered the question

A good answer might be:

$$\begin{pmatrix} -7 & 1 & -1 \\ 9 & -14 & 4 \\ -1 & 6 & 3 \end{pmatrix}$$

Real Number Elements

Of course, each element of a matrix is a scalar, a real number--in computer science terms, a floating point number. The examples use integers so the arithmetic is easy. Here is another example, this time with floating point values.



Most spreadsheet programs and medium priced electronic calculators include matrix math functions. You might wish to investigate whatever you have. But for these exercises, try to do the work "in your head" in order to internalize the process.

"It is important to practice the ... procedure for multiplying matrices until it becomes automatic. Also, you should be able to pick out immediately the row of **A** and the column of **B** that combine to give a particular entry in **AB**."[1]

1. Ben Noble and James Daniel, *Applied Linear Algebra*, 3rd., Prentice-Hall, 1988.

QUESTION 7:

Is it always true that **AB** = **BA**?

Click Here after you have answered the question

A good answer might be:

Yes.

End of the Chapter

That about wraps it up for this chapter. You may wish to review the following. Click on a blue subject that interests you to go to where it was discussed. To get back here, click on the "back arrow" button of your browser.

- Definition of a matrix.
- Matrix elements.
- Matrix dimensions.
- Names for matrices and their elements.
- When two matrices are equal.
- Matrix addition.
- The Zero matrix.
- Rules for adding matrices.
- Scalar multiplication of a matrix.
- Negation of a matrix.
- Matrix Subtraction.
- Matrix Transpose

The next chapter will discuss more things that can be done with matrices.

Click here to go back to the main menu.

You have reached the End.